Does quasi-long-range order in the two-dimensional XY model really survive weak random phase fluctuations?

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Effective theories for random critical points are usually non-unitary, and thus may contain relevant operators with negative scaling dimensions. To study the consequences of the existence of negative dimensional operators, we consider the random-bond XY model. It has been argued that the XY model on a square lattice, when weakly perturbed by random phases, has a quasi-long-range ordered phase (the random spin wave phase) at sufficiently low temperatures. We show that infinitely many relevant perturbations to the proposed critical action for the random spin wave phase were omitted in all previous treatments. The physical origin of these perturbations is intimately related to the existence of broadly distributed correlation functions. We find that those relevant perturbations do enter the Renormalization Group equations, and affect critical behavior. This raises the possibility that the random XY model has no quasi-long-range ordered phase and no Kosterlitz-Thouless (KT) phase transition.

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I. INTRODUCTION

The theory of the Renormalization Group (RG) and, in particular, the concept of relevant and irrelevant operators provides a general and deep understanding of critical points in clean systems. The physical picture at the heart of the RG theory is of sufficient generality to have been applied to the study of critical points in disordered systems such as random magnetic systems or the problem of Anderson localization [1–3].

However, effective theories for critical points induced by disorder (in short, random critical points) are usually non-unitary, i.e., they contain operators whose scaling dimensions need not be bounded from below. Thus, it is possible for a random critical point to be endowed with operators whose scaling dimensions are negative. This possibility is closely related to the fact that observables can be very broadly distributed in a critical phenomena induced by disorder.

A paradigm of this situation is the problem of a "relativistic" particle moving in two spatial dimensions in the background of a static but random vector potential, in short the random-Dirac-fermion problem [4–10]. This system was found to have a line of critical points such that each critical point contains operators with negative scaling dimensions that carry trivial quantum numbers associated to the symmetries in the problem [7,8]. In other words, those operators are relevant and may appear in the effective theory. Now the question is, do the random-Dirac-fermion critical points really exist or are they destroyed by the relevant operators? To answer this question, the following issue needs to be addressed. Do these unusual relevant operators affect the critical points in the conventional way, can we use the standard RG arguments to study the stability of random critical points? One concern is that negative dimensional operators might have some special properties that would prevent the use of standard RG arguments. For example, whereas the identity operator has zero scaling dimension, it does not destabilize a critical point since it cannot affect the scaling of any correlation functions. The question we want to address in this paper is what happens to a critical point and the corresponding correlation functions when relevant operators with negative scaling dimensions appear in the effective theory.

A second unusual property of the random-Dirac-fermion critical point relative to a generic critical point describing a clean system is that an infinite number of relevant operators appear simultaneously in the effective theory. The RG flow in the vicinity of the random-Dirac-fermion critical point must then involve infinitely many coupled equations in which case the issue of its stability becomes much more intricate.

Since there is no general principle ruling out the existence of an infinite number of relevant operators with negative scaling dimensions at a random critical point, it is imperative to reexamine the stability of random critical points with this possibility in mind. Usually, in the literature on random critical points, only the effects of a finite number of perturbing operators are investigated. The properties of the random-Dirac-fermion critical points suggest, however, that it is very important to study the scaling properties of "complex" operators. At a random-Dirac-fermion critical point, the scaling dimensions of those complex operators are of the form $h=h_0n-gn^2$, where n is an integer that characterizes the complexity of the operator and g characterizes the strength of the randomness. The trademark of these complex operators is that, whereas they become more irrelevant the more complex they are (the larger n is) in the absence of disorder, this behavior is reversed with disor-

der. In fact, since n is unbounded from above, it is seen that the random-Dirac-fermion critical points ($g > 0$) always have an infinite number of negative dimensional operators. For weak randomness, only complex operators with very large n can have negative dimensions. In early studies of random-Dirac-fermion critical points, the effects of such complex operators (strongly irrelevant in the absence of disorder but relevant in the presence of disorder) were not accounted for.

In this paper we want to gain more insights into random critical points characterized by an infinite number of operators with negative scaling dimensions, elucidate the physical origin of such operators, and study how these operators manifest themselves in the standard RG treatment of the stability of a random critical point. To this end, we will reexamine the problem of the *random bond* two-dimensional XY model

$$H_{XY} := \sum_{\langle ij \rangle} J_{ij} \left[1 - \cos\left(\phi_i - \phi_j - A_{ij}\right)\right]. \qquad (1.1)$$

Here, the angles $0 \leq \phi_i < 2\pi$ are defined on the sites i of a square lattice. The positive exchange couplings J_{ij} or spin stiffness and the real phases A_{ij} are defined on all directed nearest-neighbor pairs of sites $\langle ij \rangle$ and are independently distributed on the links $\langle ij \rangle$ with variances (mean values) g_J (J) and g_A (0), respectively.

Rubinstein et al. have studied the random bond XY model at the Gaussian level with randomness in the phase only ($g_J = 0$, $g_A > 0$) [11]. The Gaussian approximation consists in replacing Eq. (1.1) by the continuum limit

$$H_G := \frac{J}{2} \int d^2\mathbf{x} \sum_{\mu=1}^{2} \left[\partial_\mu(\varphi + \Theta) - A_\mu\right]^2, \qquad (1.2a)$$

where φ is vortex free whereas Θ carries vortices. The probability distribution for the static random vector potential A_μ is also taken to be Gaussian,

$$P[A_\mu] := \frac{\exp\left(-\frac{1}{2g_A}\int d^2\mathbf{x}\ A_\mu^2\right)}{\int \mathcal{D}[A_\mu] \exp\left(-\frac{1}{2g_A}\int d^2\mathbf{y}\ A_\mu^2\right)}, \qquad (1.2b)$$

where we adopt the summation convention over repeated indices from now on. Rubinstein et al. argue that the random bond XY model belongs to the same universality class as the Gaussian model and they infer from the Gaussian model that, for any given but sufficiently small disorder strength g_A, there exists a line of critical points ending at a KT-like transition. In other words, for fixed g_A the KT phase diagram of the pure system [12] is preserved albeit with scaling exponents depending on g_A [11]. At the heart of their argument is an estimate for the disorder average of a two-point correlation function for an operator associated with vortices. The relevance/irrelevance of this vortex operator controls the KT-like transition.

The manifold of random critical points found by Rubinstein et al. is quite special. First, these are non-trivial random critical points since scaling exponents depend both on temperature and disorder strength. This property should be contrasted with that of a critical point for which the effect of disorder is fully accounted for by irrelevant random perturbations as happens in the two-dimensional Ising model with weak bond randomness [13]. Second, each random critical point is exactly soluble, all local operators can be listed and their scaling dimensions can be calculated. Third, there is an infinity of operators associated to vortices that carry negative scaling dimensions. We want to use the Gaussian approximation to the random bond XY model as a testing ground to gain some insights about special properties of random critical points associated to a spectrum of negative scaling dimensions without lower bound.

We close this introduction by pointing out that, besides the relevance of the two-dimensional random XY model to magnetic systems with random Dzyaloshinkii-Moriya interactions [11], crystal systems on disordered substrates [14], arrays of Josephson junctions with positional disorder [15], and vortex glasses [16], the random XY model is also closely related to spectral properties of two-dimensional Dirac Hamiltonians with random vector potential and random mass. In turn, random Dirac fermions in two dimensions can be connected [17] to statistical problems such as the random flux-line model in the mixed phase of superconductors [18] and driven random diffusion model [19]. We hope that a better understanding of the random XY model might be useful to this class of problems.

II. RESULTS

It is easy to show that, if vortices are not allowed (the spin wave approximation), the continuum model Eq. (1.2) is at a fixed point where the spin $\exp(\mathrm{i}\varphi)$ has algebraic quasi-long-range correlations for any temperature and any disorder strength. This phase, the *random spin wave phase*, is an exactly soluble random critical point.

In this paper we would like to reexamine the stability of this quasi-long-range ordered phase. The stability of this phase has been studied before. If one assumes that the ground state in the vortex sector is in the dipole phase and if one considers the binding and un-binding of the simplest vortices within the first non-trivial order of a fugacity expansion for the vortices [11], one finds that the random critical point is stable only for a range of temperatures and disorder strengths bounded by the dashed line in Fig. 1. A more sophisticated approximation consists in treating a non-interacting gas of dipoles in the presence of disorder non-perturbatively in the vortex fugacity, in which case quasi-long-range order is present in the shaded area in Fig. 1 [20]. However, in this paper we find that:

1. For any temperature and any disorder strength of the random phases, randomness in the vortex fugacity generates an infinite number of relevant terms (most of them carry negative scaling dimensions) in the critical action of the *Gaussian approximation to the random bond XY model* that describes the random spin wave phase.

2. If the continuum model Eq. (1.2) is deduced from a random phase only ($g_J = 0$, $g_A > 0$) XY model on a lattice, then its vortex fugacity is necessarily random.

3. Most importantly, the above relevant terms in the critical action describing the random spin wave phase do enter the perturbative RG equations to each order in the fugacity expansion, and completely modify the RG flow at long distances. Thus, the relevant terms have the potential to cause an instability of the random spin wave phase.

Let us call the simultaneous presence of relevant terms in a critical action and in the RG equations *a perturbative instability*. Thus, we may say that the random spin wave phase of the random bond XY model has a perturbative instability for any temperature and disorder strength, if the vortex fugacity is itself random.

We would like to stress that when there are finitely many relevant terms, the perturbative instability implies the instability of the critical point. Since they enter the RG equations, the finite number of relevant perturbations either destroys the algebraic long-range correlations altogether or changes critical exponents.

However, in our case, the perturbative instability corresponds to infinitely many relevant terms appearing simultaneously in the critical action and in the RG equations. It is thus not completely clear to us what are the effects of such a perturbative instability, after one sums up an infinite number of contributions to the RG equations from all relevant operators. By contrast, if all but a finite number of relevant operators can be switched off from the critical action, these remaining perturbations would completely alter the correlation functions at long distances.

At the very least, the perturbative instability in the random bond XY model represents a new situation with regard to the issue of the stability of random critical points which must be addressed. The possibility of this new situation (i.e., the presence of an infinite number of relevant operators in the critical action) is closely related to the fact that the moments of a random variable, say the exponentiated energy of a dipole of vortices, need not be bounded if the random variable is sufficiently broadly distributed. Hence, random critical points need not be described by unitary field theories and scaling exponents need not be bounded from below.

Korshunov [21] was the first to argue that there might not be any quasi-long-range ordered phase in the random phase XY model. To this end, he introduced a sequence of local operators, $O_{r;N}(\mathbf{x})$, for the replicated random phase XY model labeled by the two index r and N. For given integer values of r and $N \leq r$, $O_{r;N}(\mathbf{x})$ creates N vortices, each belonging to a different replica of the XY model, on site $\mathbf{x}$. Here, r is the total number of replica. Korshunov found that for any strength of disorder, the scaling dimension of $O_{r;N}(\mathbf{x})$ becomes negative upon analytical continuation to the $r \downarrow 0$ limit, provided N is fixed and chosen sufficiently large.

We recall that the replica approach identifies a given physical operator Q with a family or sequence of operators Q_r labeled by the total number of replicas r. In this paper, we identify the family of operators $O_{r;N}$ (labeled by r) studied by Korshunov with a physical operator. More precisely, we show how the family of operators $O_{r;N}$ can be induced in a physical way in the effective action describing the random spin wave phase.

In the absence of vortices, the disorder average over the two-point function of a local operator has the form $A|\mathbf{x} - \mathbf{y}|^{\alpha}$ in the random bond XY model. On the one hand, if the fugacity expansion is valid (as is the case in the clean XY model), then the exponent α and the coefficient A depend on the fugacity Y and have an analytic expansion around $Y = 0$. On the other hand, the breakdown of the fugacity expansion can have three consequences:

1. $A(Y)$ is not analytic around $Y = 0$, but $\alpha(Y)$ is.

2. Both $A(Y)$ and $\alpha(Y)$ are not analytic around $Y = 0$.

3. The critical behavior is completely changed or destroyed by the inclusion of vortices.

By extending the Renormalization Group (RG) equations to fourth order in the fugacity, we can show that the fugacity expansion breaks down according to scenario 2, i.e., *both* the coefficient $A(Y)$ and the scaling exponent $\alpha(Y)$ are not analytic functions of the vortex fugacity. Furthermore, if conventional RG arguments apply, one may then conclude that vortices change or destroy the critical line of the the random XY model for any temperature and any impurity strength according to scenario 3.

The paper is organized as follows. We first show in section III that the two-point correlation function studied in [11] to construct the phase diagram has a very broad probability distribution, a fact expressed in the underlying critical theory describing the random spin wave phase by the presence of infinitely many negative dimensional operators.

We then show that the random spin wave phase is perturbatively unstable. Indeed, this instability manifests itself by the non-analyticity of the fugacity expansion for disorder averaged correlation functions due to the existence of infinitely many negative dimensional operators in the underlying critical theory. A summary of our arguments is presented in section IV with technicalities relegated to appendix B.

Another signature of the perturbative instability, as we show in section V, is illustrated by the fact that any randomness in the spin stiffness of the random bond XY model induces infinitely many relevant perturbations to the critical theory describing the random spin wave phase in the Gaussian approximation.

The relationship between the random bond XY model, the random bond Villain model, and the Gaussian approximation is discussed in section VI. It is pointed out that the fugacity expansion already breaks down in the random bond Villain model.

Our conclusions are followed up by two appendices. The first one discusses ground state properties and some probability distributions for correlation functions are calculated. The second one presents a detailed derivation of the perturbative RG analysis up to fourth order in the fugacity expansion. There it is also shown that there exists a one to one correspondence between correlation functions for the perturbations induced by a random fugacity within the replica approach of section V, and the contributions to the fugacity expansion of section IV.

III. BROADLY DISTRIBUTED CORRELATION FUNCTIONS

In this section we construct the random spin wave phase and show that it is described by a manifold of random critical points. The central quantity of interest in the spin wave phase is the thermal correlation function for two spins. We characterize uniquely its probability distribution by all its moments. All moments are well-behaved.

We then turn our attention to the vortex sector and, in particular, to the exponentiated energy of a pair of vortices of opposite charges (a dipole) in the background of the random vector potential. All moments of this exponentiated energy are calculated in Eq. (3.18). This is the central result of this paper. In contrast to the spin wave sector, higher moments dominate the lower ones. This property is nothing but the signature of a log-normal distribution for the exponentiated dipole energy. Correspondingly, the energy of a dipole is Gaussian distributed in model (1.2). Conversely, the critical theory describing the random spin wave phase *must* contain infinitely many operators with negative scaling dimensions that are associated with vortices in order to account for the Gaussian distribution of the dipole energy.

A. Factorization into a spin wave and a vortex sector

We begin with the model in the continuum defined by Eq. (1.2a). We *reiterate* that A_μ is assumed Gaussian distributed with variance g_A, i.e., that

$$P[A_\mu] \propto \exp\left(-\frac{1}{2g_A}\int d^2\mathbf{x}\ A_\mu^2\right). \tag{3.1}$$

The justification for this assumption is that the precise shape of the probability distribution should leave critical properties unchanged as long as the probability distribution preserves the short-range nature of spatial correlations in the disorder. In particular, the tails of the probability distribution of *non-compact support* in Eq. (3.1) should not affect critical properties.

To understand the role of the random vector potential, it is convenient to decompose it into transverse and longitudinal components:

$$A_\mu = \tilde\partial_\mu\theta + \partial_\mu\eta, \quad \tilde\partial_\mu := \epsilon_{\mu\nu}\partial_\nu. \tag{3.2}$$

The advantage of this decomposition is that the partition function derived from $H_G = \int d^2\mathbf{x}\mathcal{H}_G$ in Eq. (1.2a) and the probability distribution in Eq. (3.1) both factorize since

$$\mathcal{H}_G = \frac{J}{2}\{[\partial_\mu(\varphi-\eta)]^2 + [\tilde\partial_\mu(\tilde\Theta-\theta)]^2\}, \tag{3.3}$$

$$P[\theta,\eta] \propto \exp\left\{-\frac{1}{2g_A}\int d^2\mathbf{x}\left[(\partial_\mu\theta)^2+(\partial_\mu\eta)^2\right]\right\}. \tag{3.4}$$

Here, $\tilde\Theta$ is dual to Θ [1], and one must implement the constraint on the disorder that there be no zero modes:

$$\int d^2\mathbf{x}\ \theta(\mathbf{x}) = 0, \qquad \int d^2\mathbf{x}\ \eta(\mathbf{x}) = 0. \tag{3.5}$$

The ambiguity in the choice of the integration constant in Eq. (3.2) is thus removed.

B. The spin wave sector

The consequences of the random vector potential on the spin wave sector are trivial. One can shift spin wave integration variables to

$$\varphi' := \varphi - \eta. \tag{3.6}$$

In the absence of randomness in the spin stiffness, all correlation functions for $\exp(\mathrm{i}\varphi)$=$\exp(\mathrm{i}\varphi' + \mathrm{i}\eta)$ can be

[1] Given Θ, the dual $\tilde\Theta$ is defined by $\partial_\mu\Theta = \tilde\partial_\mu\tilde\Theta$. For example, if $\Theta = \sum_{i=1}^M m_i \arctan\left[\frac{(\mathbf{x}-\mathbf{x}_i)_2}{(\mathbf{x}-\mathbf{x}_i)_1}\right]$, then $\tilde\Theta = -\sum_{i=1}^M m_i \ln\left|\frac{\mathbf{x}-\mathbf{x}_i}{l_0}\right|$, where the vorticities m_i are integer and l_0 is an arbitrary length scale.

calculated. In turn, the disorder average over η can be performed since the probability distribution for η is that of a free scalar field in two dimensions.

For example,

$$\begin{aligned}\overline{\langle e^{\mathrm{i}\varphi(\mathbf{y}_1)}\ e^{-\mathrm{i}\varphi(\mathbf{y}_2)}\rangle_q} &\propto \frac{\overline{e^{\mathrm{i}q\eta(\mathbf{y}_1)}\ e^{-\mathrm{i}q\eta(\mathbf{y}_2)}}}{|\mathbf{y}_1-\mathbf{y}_2|^{\frac{q}{2\pi K}}} \\ &= \frac{1}{|\mathbf{y}_1-\mathbf{y}_2|^{\frac{q+g_A K q^2}{2\pi K}}}. \end{aligned} \tag{3.7}$$

Thermal averaging is denoted by angular brackets. Disorder averaging is denoted by an overline and induces a quadratic dependency on the moment q for the scaling exponent. Thus, the impact of the quenched random vector potential on the spin wave sector is to drive the system to a new critical point for any strength of the disorder g_A and for any reduced spin stiffness

$$K := J/T. \tag{3.8}$$

The random vector potential is seen to destroy the long-range order at vanishing temperature by replacing it with quasi-long-range order. For all finite temperatures, the algebraic decays of the spin correlation functions are more pronounced due to the disorder. On the other hand, random spin stiffness remains irrelevant since it amounts to a random temperature (more formally, one verifies that, for any integer valued $q > 0$, $(\partial_\mu\varphi')^{2q}$ is a strongly irrelevant operator everywhere along the spin wave critical line $K \geq 0$). Note that this argument is nothing but Harris criterion [22] in disguise. Finally, by choosing $|\mathbf{y}_1 - \mathbf{y}_2|$ sufficiently large, the two-point function $\langle\exp[\mathrm{i}\varphi(\mathbf{y}_1)-\mathrm{i}\varphi(\mathbf{y}_2)]\rangle$ is seen to be a random variable with an arbitrarily small random component. This is not so on all counts in the vortex sector.

C. The vortex sector

Vortices in the XY model are described by the field Θ. More precisely, the local density of vortices on the Euclidean plane is given by $\partial_\mu^2\Theta$. Only the component $\tilde{\partial}_\mu\theta$ of the random vector potential A_μ couples *directly* to the vortices described by Θ. Whereas the field Θ is induced by integer valued vortices, the quenched disorder $\tilde{\partial}_\mu\theta$ describes real valued vortices. Hence, the system tries to minimize the energy by screening the real valued quenched vortices with thermally excited integer valued vortices. However, by doing so, entropy is lost. The balance of energy and entropy could lead to a KT-like critical temperature separating a low temperature phase with positive free energy and a high temperature phase with negative free energy.

In fact in the absence of randomness in the spin stiffness, the existence of a KT transition is suggested by a perturbative RG calculation in the Coulomb (CB) gas representation

$$\begin{aligned} S_{\rm CB}[\Theta,\theta] := {} & E\sum_k (m_k-n_k)^2 \\ & -\pi K\sum_{k\neq l}(m_k-n_k)(m_l-n_l)\ln\left|\frac{\mathbf{x}_k-\mathbf{x}_l}{l_0}\right| \end{aligned} \tag{3.9}$$

of Eq. (3.3), provided g_A is not too large and *assuming* the existence of a dipole phase of the CB gas at sufficiently low temperature and large reduced bare vortex core energy E [11,20]. Again, Θ is induced by a neutral configuration of vortices with vorticities $m_k \in \mathbf{Z}$, whereas θ is induced by a neutral configuration of vortices with vorticities $n_l \in \mathbf{R}$. It is sufficient to consider neutral configurations since the energy cost of creating net vorticity scales logarithmically with the system size L, l_0 being an arbitrary length scale.

The perturbative RG analysis in the CB gas representation is usually summarized by the phase diagram of Fig. 1. The phase diagram is three dimensional with $1/K = T/J$ the dimensionless temperature, g_A measuring the disorder strength, and $Y_1 = \exp(-E)$ the fugacity of charge one vortex. *All points on the plane with vanishing fugacity are critical.* This is the manifold of critical points describing the random spin wave phase. Critical points within the shaded area are argued to be stable [11,20], i.e., Y_1 is irrelevant and thus decreases at long distances. Critical points outside the shaded area are unstable, i.e., Y_1 is relevant and thus grows at long distances.

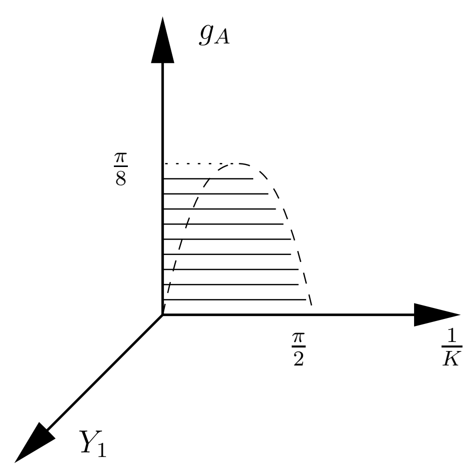

FIG. 1. Proposed phase diagram for CB gas with quenched randomly fractionally charged vortices. $1/K$ is the reduced temperature, g_A the variance of the Gaussian disorder $\tilde{\partial}_\mu\theta$, Y_1 the charge one fugacity for thermal vortices.

To go beyond these results (still keeping J nonrandom), we prefer the Sine-Gordon (SG) representation of the CB gas in Eq. (3.9) with $m_k = \pm 1$ only. By an expansion in powers of the "magnetic field" h_1 of the Boltzmann weight with Lagrangian

$$\mathcal{L}_{\rm SG}[\chi,\theta] := \frac{1}{2t}(\partial_\mu\chi)^2 - \frac{h_1}{t}\cos\chi + \frac{\rm i}{2\pi}\chi(\partial_\mu^2\theta), \tag{3.10}$$

followed by an integration over χ, we recover the grand canonical partition function of the charge one CB gas derived from Eq. (3.9) provided one identifies

$$K = \frac{t}{4\pi^2}, \quad Y_1 \sim \frac{h_1}{2t}, \quad \theta(\mathbf{x}) = \sum_{l=1}^{N} n_l \ln\left|\frac{\mathbf{x}-\mathbf{y}_l}{l_0}\right|. \tag{3.11}$$

Notice that χ couples to the disorder $\tilde{\partial}_\mu\theta$ through a purely imaginary coupling, and that higher charges vortices $m_k = \pm 2, \cdots$ are easily incorporated with higher harmonic $\cos(k\chi)$, $k \in \mathbf{N}$. Hence, were it not for the "magnetic field" h_1, χ and θ would decouple after the shift of integration variable

$$\chi =: \chi' + \frac{{\rm i}t}{2\pi}\theta \tag{3.12}$$

very much in the same way the spin waves decouple from the longitudinal realizations of the disorder.

In the absence of disorder, one can establish the existence of the KT transition by performing a perturbative RG analysis on the two-point function

$$\begin{aligned}\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle_0 :=& \frac{\int \mathcal{D}[\chi]\ e^{-\int d^2\mathbf{x}\mathcal{L}_{\rm SG}[\chi,0]}\ e^{{\rm i}\chi(\mathbf{x}_1)-{\rm i}\chi(\mathbf{x}_2)}}{\int \mathcal{D}[\chi]\ e^{-\int d^2\mathbf{x}\mathcal{L}_{\rm SG}[\chi,0]}}\\ \equiv& \frac{\int \mathcal{D}[\chi]\ e^{-S_{\rm SG}[\chi,0]}\ e^{{\rm i}\chi(\mathbf{x}_1)-{\rm i}\chi(\mathbf{x}_2)}}{\mathcal{Z}_{\rm SG}[0]}.\end{aligned} \tag{3.13}$$

In short, one first expands the right hand side of Eq. (3.13) in powers of a very small fugacity $h_1/2t$. Without a short distance cutoff a, all coefficients of the expansion in the fugacity are ill-defined. The arbitrariness in the choice of the short distance cutoff is used to derive RG equations obeyed by the fugacity and the reduced temperature. The RG equations are integrated to determine whether the initial assumption of a very small fugacity is consistent. The irrelevance, marginality, and relevance of the fugacity then determines the spin wave phase, KT transition, and disordered phase of the XY model, respectively.

Rubinstein et al. [11] followed the same strategy in the presence of the quenched vector potential $\tilde{\partial}_\mu\theta$. More precisely, they performed a RG analysis of the fugacity expansion of two correlation functions:

$$G_{\mathbf{x}_1,\mathbf{x}_2} := -\ln\overline{\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle}, \tag{3.14}$$

$$\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle := \frac{\int \mathcal{D}[\chi]\ e^{-S_{\rm SG}[\chi,\theta]}\ e^{{\rm i}\chi(\mathbf{x}_1)-{\rm i}\chi(\mathbf{x}_2)}}{\mathcal{Z}_{\rm SG}[\theta]}, \tag{3.15}$$

to the first non-trivial order in the fugacity. Notice that it is necessary to include both $G_{\mathbf{x}_1,\mathbf{x}_2}$ and $\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle$ to close the RG equations to the first non-trivial order in the fugacity. This is not surprising since taking the logarithm does not usually commute with averaging.

The crucial point of the fugacity expansions in Eqs. (3.14,3.15) is that every expansion coefficients depend on correlation functions calculated for vanishing "magnetic field" h_1 (fugacity $h_1/2t$) such as

$$\int d^2\mathbf{y}_1\cdots d^2\mathbf{y}_{2n}\overline{\langle e^{{\rm i}[\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)+\chi(\mathbf{y}_1)+\cdots-\chi(\mathbf{y}_{2n})]}\rangle_{h_1=0}}, \tag{3.16}$$

on the one hand, but also such as

$$\overline{\langle e^{{\rm i}[\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\rangle_{h_1=0}\left\langle\int d^2\mathbf{y}_1 d^2\mathbf{y}_2 e^{{\rm i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_2)]}\right\rangle^n_{h_1=0}}, \tag{3.17}$$

on the other hand. For vanishing fugacity $h_1/2t$, all averaged correlation functions are algebraic and in particular [compare with Eq. (3.7)]

$$\begin{aligned}\overline{\langle e^{{\rm i}\chi(\mathbf{y}_1)}\ e^{-{\rm i}\chi(\mathbf{y}_2)}\rangle^q_{h_1=0}} \propto& \frac{\overline{e^{-\frac{qt}{2\pi}\theta(\mathbf{y}_1)}\ e^{\frac{qt}{2\pi}\theta(\mathbf{y}_2)}}}{|\mathbf{y}_1-\mathbf{y}_2|^{\frac{qt}{2\pi}}}\\ =& \frac{1}{|\mathbf{y}_1-\mathbf{y}_2|^{2\pi Kq(1-g_A Kq)}}.\end{aligned} \tag{3.18}$$

This is our most important result. We will make use of it in section IV to deduce that the fugacity expansion cannot yield analytic scaling exponents around vanishing fugacity. The remarkable property of Eq. (3.18) is that the scaling exponent becomes negative for any given temperature $1/K$ and disorder strength g_a as long as the moment q is sufficiently large. Hence, there must exist infinitely many local operators with negative scaling dimensions in the effective theory describing the random spin wave phase.

The correlation function in Eq. (3.18) is closely related to the strength of the interaction between two external charges in the CB gas. Indeed, as we show in appendix A,

$$\langle e^{{\rm i}\chi(\mathbf{y}_1)}\ e^{-{\rm i}\chi(\mathbf{y}_2)}\rangle_{h_1=0} = e^{-\frac{K}{J}H_{1,2}}. \tag{3.19}$$

Here, $H_{1,2}$ is the bare (since $h_1 = 0$) energy of two vortices of opposite unit charges in the background of the vector potential $\tilde{\partial}_\mu\theta$. The probability distribution of $H_{1,2}$ is calculated in appendix A and shown to be a Gaussian distribution with a variance growing logarithmically with $|\mathbf{y}_1-\mathbf{y}_2|$. Hence, the random variable $\exp[(K/J)H_{1,2}]$ has a log-normal distribution. For a fixed separation $|\mathbf{y}_1-\mathbf{y}_2|$, the random energy $H_{1,2}$ can take arbitrarily negative values as a consequence of our initial assumption on the probability distribution in Eq. (3.1). This fact explains why the random variable $\langle\exp[{\rm i}\chi(\mathbf{y}_1)-{\rm i}\chi(\mathbf{y}_2)]\rangle$ is unbounded from above.

Correspondingly, the ratio of the q moment to the first one raised to the power q grows with $|\mathbf{y}_1-\mathbf{y}_2|$ raised to the positive power $+2\pi g_A K^2 q(q-1)$, in sharp contrast

to the moments of the logarithm of correlation functions [see Eq. (A10)] on the one hand, or to the moments of correlation functions in the spin wave sector on the other hand.

We infer from Eq. (3.18) that a sufficiently large moment of the two-point function in Eq. (3.18) is not bounded from above for arbitrary large values of the separation $|\mathbf{y}_1 - \mathbf{y}_2|$. This property is a consequence of $H_{1,2}$ being Gaussian distributed that can also be understood as follows. On the second line of Eq. (3.18), the disorder average is dominated by realizations of the disorder with $\theta(\mathbf{y}_1)$ very negative and $\theta(\mathbf{y}_2)$ very positive. Such configurations are extremely rare for small separation $|\mathbf{y}_1 - \mathbf{y}_2|$, since the cost $(\tilde{\partial}_\mu \theta)^2$ will then be substantial, but they become more likely as $|\mathbf{y}_1 - \mathbf{y}_2|$ increases.

The sign of the q^2 dependency of the scaling exponents $2\pi K q\,(1 - g_A K q)$ on the right hand side of Eq. (3.18) is thus the signature of broadly distributed random variables. *We will show below that operators with the same scaling dimensions appear in the replicated action if random spin stiffness is introduced.* The same scaling exponents have also been found by Korshunov to control the fugacity expansion of correlation functions such as $\overline{\langle F_{\mathbf{x}_1,\mathbf{x}_2} \rangle}$ to order $2q$ [21] (see section IV and appendix B). Korshunov concluded from this property of the fugacity expansion that any quasi-long-range ordered phase should be destroyed for arbitrary weak random vector potential $\tilde{\partial}_\mu \theta$. An alternative conclusion, however, is that quasi-long-range order is characterized by scaling exponents that are non-analytic functions of the fugacity. We will come back to this scenario in section IV and in the appendices.

The parabola

$$g_A^{(1)}\left(\frac{1}{K}\right) := \frac{1}{K}\left(1 - \frac{2}{\pi}\frac{1}{K}\right) \tag{3.20}$$

(dashed line in Fig. 1) is obtained from the first moment $q = 1$ by requiring that the scaling exponent on the right hand side of Eq. (3.18) be "marginal", i.e., equals 4 [11]. It is argued to delimit the boundary between quasi-long-range order and disordered phase for $\pi/4 \leq 1/K \leq \pi/2$ [11].

Instead of the reentrant phase transition for $0 \leq 1/K \leq \pi/4$, the dilute vortex pair approximation proposed in [20] suggests that the parabola should be replaced by the dotted line in Fig. 1. The difficulties with the fugacity expansion are here bypassed altogether since it is possible to calculate the mean of $G_{\mathbf{x}_1,\mathbf{x}_2}$ non-perturbatively in the fugacity provided it is assumed that: (i) *an insulating dipole phase exists* and (ii) *the interaction between dipoles can be neglected.* Nevertheless, it remains an open problem to show rigorously that at $T = 0$ and for infinitesimal g_A, the ground state configuration is in some quasi-long-range ordered phase consistent with assumptions (i) and (ii) (see appendix A).

IV. NON-ANALYTICITY OF THE FUGACITY EXPANSION

We are now ready to describe the results obtained from the fugacity expansion of correlation functions in the SG model Eq. (3.10). We will prove that the fugacity expansion is non-analytic. For the pure system, non-analyticity of the fugacity expansion is interpreted as the destruction of quasi-long-range order. In the presence of a random vector potential, we cannot rule out the possibility that an exotic phase survives with quasi-long-range order characterized by scaling exponents which are non-analytic functions of the fugacity. However, even if a quasi-long-range ordered phase is present in the phase diagram of the SG partition function, we will show in section V that randomness in the spin stiffness induces infinitely many relevant perturbations to this critical behavior.

The mathematical reason for the breakdown of the fugacity expansion is that we are expanding a random function in powers of a random variable that takes values outside the radius of convergence of the expansion. The physical reason for the breakdown is that the typical ground state of the random phase XY model does not support long-range-order (the ferromagnetic state). For the KT transition to survive the presence of a weak random vector potential $\tilde{\partial}_\mu \theta$, the typical ground state must contain a sufficiently large number of tightly bound pairs of vortices so as to destroy long-range order, but not sufficiently large so as to screen the bare logarithmic interactions of the vortices (see appendix A). Since the vortex fugacity measures, to a first approximation, the density of vortices, a ground state with quasi-long-range order must imply the breakdown of a fugacity expansion around vanishing fugacity (the ferromagnetic ground state).

Although the fugacity expansion is non-analytic, it is still useful to decide if non-analyticity reflects only that of scaling exponents of algebraic correlation functions or if it signals the breakdown of algebraic order. We have performed the fugacity expansion in the SG representation to fourth order in the fugacity and could not, to this order, distinguish between an exotic algebraic phase from the complete destruction of quasi-long-range order.

Lastly, the fugacity expansion is also instructive in that it allows to classify and understand the role played by the large numbers of local operators that can be constructed within the replica formalism. The close relationship between the fugacity expansion and the replica formalism will be established below together with appendix B and section V.

In principle, we would like to calculate the probability distribution of the two-point functions $G_{\mathbf{x}_1,\mathbf{x}_2}$ and $\langle F_{\mathbf{x}_1,\mathbf{x}_2} \rangle$. This is done in appendix A for vanishing fugacity. For finite fugacity, we are only able to calculate their moments perturbatively in the fugacity. We expect $G_{\mathbf{x}_1,\mathbf{x}_2}$ to be close to a Gaussian distribution since

it is already Gaussian distributed for vanishing fugacity. Hence, we will only calculate the mean of $G_{\mathbf{x}_1,\mathbf{x}_2}$. On the other hand, we will need all moments of $\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle$ since it is log-normal distributed for vanishing fugacity. Our goal is thus to calculate perturbatively in powers of the bare fugacity $h_1/2t$

$$\overline{G_{12}} := \overline{G_{\mathbf{x}_1,\mathbf{x}_2}}, \tag{4.1}$$

$$\overline{\langle F_{12}\rangle^q} := \overline{\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle^q}, \quad q \in \mathbf{N}. \tag{4.2}$$

We restrict configurations of (thermal as well as quenched) vortices to neutral ones. This implies that only even powers of the bare fugacity $h_1/2t$ enter in Eqs. (4.1,4.2). The calculation to lowest order in the fugacity is summarized by Eq. (3.18).

A. Fugacity expansion up to second order

Up to second order in the bare fugacity $h_1/2t$, we find (see appendix B) that the mean of the two-dimensional CB interaction between two external charges of opposite sign is

$$\overline{G_{12}} \approx 2\pi \left[K - 4\pi^3 K^2 Y^2_{(1;1)} \int_1^{L/a} dy y^{3-2\pi\bar{K}}\right] \ln\left|\frac{\mathbf{x}_{12}}{a}\right|, \tag{4.3}$$

whereas

$$\overline{[\langle F_{12}\rangle]^q} \approx \left|\frac{\mathbf{x}_{12}}{a}\right|^{-2\pi x(q)}, \tag{4.4}$$

with the scaling exponents

$$x(q) := \overline{K(q)} - 4\pi^3 q \left\{\left[\frac{\overline{K(q)}}{q}\right]^2 - K^4 g_A^2 q^2\right\} Y^2_{(1;1)} \int_1^{L/a} dy y^{3-2\pi\bar{K}}. \tag{4.5}$$

Here, we have introduced

$$Y^2_{(1;1)} := \left(\frac{a^2 h_1}{2t}\right)^2, \tag{4.6}$$

$$\bar{K} := \overline{K(1)}, \quad \overline{K(q)} := Kq - K^2 g_A q^2. \tag{4.7}$$

The reason for which we label the dimensionless fugacity $Y_{(1;1)}$ by the subindex $(1;1)$ will become clear to fourth order in the fugacity expansion. Suffices to say that to second order in the fugacity expansion, only one pair of vortices [of the type $(1;1)$] renormalizes the moments of two-point functions.

The short distance (dimensionless) cutoff in the ubiquitous integral on the right hand sides of Eqs. (4.3,4.5) is arbitrary. By splitting the range of integration into two ranges $[1, e^l[$ and $[e^l, \infty[$ with $0 < l \ll 1$, it can be shown that Eqs. (4.3, 4.5) are form invariant provided $Y_{(1;1)}$ is renormalized multiplicatively, and $K, \overline{K(q)}$ are renormalized additively. The disorder strength g_A is left unchanged in this scheme. We thus recover the well known RG equations [11]

$$\frac{dY_{(1;1)}}{dl} = (2 - \pi\bar{K})Y_{(1;1)}, \tag{4.8}$$

$$\frac{dK}{dl} = -4\pi^3 K^2 Y^2_{(1;1)}, \tag{4.9}$$

$$\frac{d\overline{K(q)}}{dl} = -4\pi^3 q \left\{\left[\frac{\overline{K(q)}}{q}\right]^2 - K^4 g_A^2 q^2\right\} Y^2_{(1;1)}, \tag{4.10}$$

$$\frac{dg_A}{dl} = 0. \tag{4.11}$$

So far so good, the scaling exponent $\bar{K}$ that controls the algebraic decay of the two-point function $\overline{\langle F_{12}\rangle_{h_1=0}}$ gives us a criterion for the breakdown of the fugacity expansion. All scaling exponents $x(q)$ are finite as long as $4 - 2\pi\bar{K} \le 0$ up to second order in the bare fugacity $h_1/2t$. This is not true anymore to fourth order in the bare fugacity.

Let us stress a few important points.

1. $Y_{(1,1)}$ appears in the RG equation of K. The latter controls the scaling dimensions of many operators.
2. When $4 - 2\pi\bar{K} < 0$, $Y_{(1,1)}$ flows to zero. The RG flow of K $[\overline{K(q)}]$ only changes K $[\overline{K(q)}]$ by a finite amount proportional to $Y^2_{(1;1)}$. Thus, scaling exponents only receive corrections of order $Y^2_{(1;1)}$. We may then say that the scaling exponents are analytic in $Y_{(1;1)}$ around $Y_{(1;1)} = 0$.
3. When $4 - 2\pi\bar{K} > 0$, $Y_{(1,1)}$ flows to infinity at long distances and K blows up. This was interpreted as the instability of the random spin wave critical point in Ref. [11]. At the very least, scaling exponents are not analytic in $Y_{(1;1)}$ around $Y_{(1;1)} = 0$ since the perturbative RG flow breaks down.

B. Fugacity expansion up to fourth and higher order

We have performed the fugacity expansion of Eqs. (4.1,4.2) to fourth order in the fugacity $h_1/2t$. The fourth order calculation shows that, *due to the disorder*, the most singular coefficients in the fugacity expansions of $\overline{G_{12}}$ and $\overline{\langle F_{12}\rangle^q}$ are proportional to the integral

$$\lim_{\frac{L}{a}\uparrow\infty} \int_1^{\frac{L}{a}} dy y^{3-2\pi\overline{K(2)}}. \tag{4.12}$$

Our derivation of the fugacity expansion of $\overline{\langle F_{12}\rangle}$ is sketched in appendix B. It is also shown there that, in general, the most singular coefficient of the fugacity expansion is proportional to the integral

$$\lim_{\frac{L}{a}\uparrow\infty}\int_1^{\frac{L}{a}} dy y^{3-2\pi\overline{K(n)}} \tag{4.13}$$

to the order $2n$. Hence, to the $2n$-th order, the regime of analyticity of the fugacity expansion is delimited by the curve [compare with Eq. (3.20)]:

$$\min\left\{g_A^{(1)}(1/K),\cdots,g_A^{(n)}(1/K):=\frac{1}{Kn}\left(1-\frac{2}{\pi}\frac{1}{Kn}\right)\right\} \tag{4.14}$$

in the plane of vanishing fugacity in the three dimensional coupling space depicted in Fig. 2.

We must therefore conclude that the fugacity expansions of Eqs. (4.1,4.2) are non-analytic for any infinitesimal value of the disorder strength g_A, since the region of analyticity in Fig. 2 shrinks to each new order and collapses to the segment $0\le 1/K\le\pi/2$, $g_A=0$ as $n\uparrow\infty$. Notice that this argument breaks down in the pure system where

$$\lim_{g_A\downarrow 0}\overline{K(n)}=Kn. \tag{4.15}$$

This is nothing but the statement that $\cos(n\chi)$ is the more irrelevant the larger n is. To put it differently, it is necessary that vortices be more irrelevant the higher their charges for the fugacity expansion to be analytic.

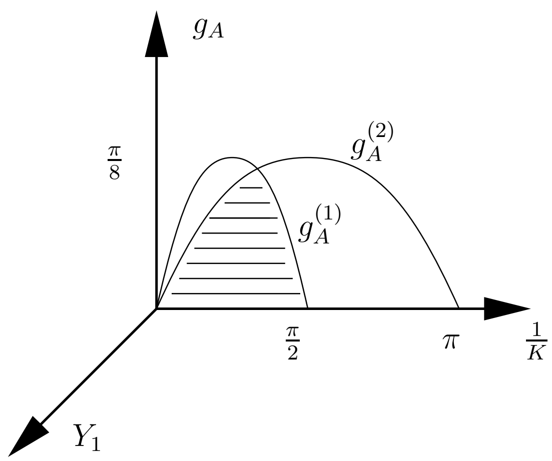

FIG. 2. Boundaries $g_A^{(1)}(1/K)$ and $g_A^{(2)}(1/K)$ in the plane of vanishing fugacity. The shaded area represents the regime of analyticity of the fugacity expansion to fourth order in the fugacity.

The integral in (4.12) appears in the RG flow to fourth order in the fugacity expansion. Indeed, we find that:

1. The RG equation of $\bar{K}$ contains a *new* fugacity $Y_{(1,1;2)}$.
2. The RG equation for $Y_{(1,1;2)}$ is

$$\frac{dY_{(1,1;2)}}{dl}=\left[2-\pi\overline{K(2)}\right]Y_{(1,1;2)}. \tag{4.16}$$

Therefore, when $4-2\pi\overline{K(2)}>0$, $Y_{(1,1;2)}$ flows to infinity and the RG flow of $\bar{K}$ blows up, suggesting the new phase boundary $2\pi\overline{K(2)}=4$, or, equivalently, $g_A^{(2)}(1/K)$. We believe that a similar structure appears to the order $2n$ in the fugacity expansion:

1. The RG equation of $\bar{K}$ contains a *new* fugacity $Y_{(1,\cdots,1;n)}$.
2. The RG equation for $Y_{(1,\cdots,1;n)}$ is

$$\frac{dY_{(1,\cdots,1;n)}}{dl}=\left[2-\pi\overline{K(n)}\right]Y_{(1,\cdots,1;n)}. \tag{4.17}$$

The quasi-long-range phase boundary is controlled by $2\pi\overline{K(2n)}=4$, or, equivalently, $g_A^{(n)}(1/K)$. A stable quasi-long-range ordered phase can only exist if $2\pi\overline{K(2n)}>4$ for all $n\in\mathbf{N}$.

Based on the conventional RG analysis which essentially assumes that one can switch off all fugacities but $Y_{(1,\cdots,1;n)}$ to order $2n$, we conclude that the quasi-long-range ordered phase is destroyed by any amount of disorder. A more conservative conclusion that we can draw is that the perturbative RG flow must break down beyond some critical order in the fugacity expansion that depends on the strength of the disorder. In any case, *scaling exponents* cannot be analytic functions of the bare vortex fugacity Y_1.

V. THE PERTURBATIVE INSTABILITY

We have shown in section III that the critical theory describing the random spin wave phase must contain an infinite number of operators with negative scaling dimensions. It was shown in section IV that these operators with negative scaling dimensions have dramatic consequences on the RG equations within the fugacity expansion. We now complete the proof of the existence of a perturbative instability by showing that the effective action from which disorder averaged correlation functions are built is necessarily perturbed by infinitely many operators with negative scaling dimensions.

The SG representation makes it clear that random spin stiffness cannot be dismissed as irrelevant as was the case in the spin wave sector since it induces a random fugacity in addition to a random temperature. Random spin stiffness has two very different consequences from a symmetry point of view.

To see this, we fermionize the SG model. We use the (Euclidean) bosonization rules [23]

$$
\begin{aligned}
&\bar{\psi}\mathrm{i}\gamma_\mu\partial_\mu\psi \to \frac{1}{8\pi}(\partial_\mu\chi)^2,\\
&\bar{\psi}\gamma_\mu\psi \to -\frac{\mathrm{i}}{2\pi}\tilde{\partial}_\mu\chi,\\
&-\mathrm{i}m_1\bar{\psi}\psi \to -\frac{h_1}{t}\cos\chi,
\end{aligned}
\tag{5.1}
$$

to relate bilinears of Grassmann variables $\bar{\psi},\psi$ to the real scalar field χ of Eq. (3.10). Here, γ_μ are any two of the three Pauli matrices. Thus, the grand canonical partition function of the CB gas derived from Eq. (3.9) is equivalent to first expanding the partition function with Thirring Lagrangian

$$
\mathcal{L}_{\mathrm{Th}}:=\bar{\psi}(\mathrm{i}\gamma_\mu\partial_\mu+\mathrm{i}m_1)\psi-\frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2+(\bar{\psi}\gamma_\mu\psi)(\tilde{\partial}_\mu\theta)
\tag{5.2}
$$

in powers of the mass $\mathrm{i}m_1$ and then integrating over $\bar{\psi},\psi$. The couplings $g,\mathrm{i}m_1$ are related to the reduced spin stiffness K by

$$
g:=\frac{1}{K}-\pi,\quad \mathrm{i}m_1\propto\frac{h_1}{4\pi^2K}.
\tag{5.3}
$$

Randomness in the vector potential of the XY model thus enters the Thirring model through the coupling of the current $j_\mu:=\bar{\psi}\gamma_\mu\psi$ of Dirac fermions to the transverse component $\tilde{\partial}_\mu\theta$ of the vector potential A_μ. Randomness in the XY spin stiffness enters the Thirring model through a random mass and through a random current-current interaction. Moreover, the random mass $\mathrm{i}m_1$ and random Thirring interaction g are generally not Gaussian distributed (even for a Gaussian distributed spin stiffness).

The fermionization of the random CB gas shows that random spin stiffness has two very different consequences from a symmetry point of view. On the one hand it induces a random mass which breaks the chiral symmetry of the kinetic energy of the Dirac fermions. On the other hand it induces a random current-current interaction which preserves the chiral symmetry. Hence, any RG calculation should renormalize $\mathrm{i}m_1$ very differently from g.

In fact, it is sufficient to ignore randomness in g altogether for two reasons, provided we include randomness in $\mathrm{i}m_1$ and $\tilde{\partial}_\mu\theta$. Firstly, randomness in g resembles non-Gaussian distributed vector potential $\tilde{\partial}_\mu\theta$. It is then straightforward to show that non-Gaussian randomness in $\tilde{\partial}_\mu\theta$ is irrelevant by calculating the scaling dimension of $(\bar{\psi}\gamma_\mu\psi)^{2q}\sim(\partial_\mu\chi)^{2q}$ on the critical plane of Fig. 1. Secondly, random $\mathrm{i}m_1$ is induced by a random fugacity of the CB gas (random magnetic field of the SG model) which will be seen to play a key role beyond the Villain approximation of the random phase XY model. Hence, it is meaningful to treat $\mathrm{i}m$, g and $\tilde{\partial}_\mu\theta$, as independent random fields and to assume that only randomness in $\mathrm{i}m$ (or h_1 in SG model) and $\tilde{\partial}_\mu\theta$ are present.

To understand the interplay between a random fugacity Y_1 (magnetic field h_1) and a random vector potential $\tilde{\partial}_\mu\theta$, we use the replica formalism based on the identity $\ln x=\lim_{r\downarrow 0}(x^r-1)/r$. Indeed, within the replica approach, disorder average can be performed directly on the replicated partition function.

Thus, starting from the replicated Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SG}}=&\sum_{a=1}^{r}\left[\frac{1}{2t}(\partial_\mu\chi_a)^2-\frac{h_1}{t}\cos\chi_a+\frac{\mathrm{i}}{2\pi}\chi_a(\partial_\mu^2\theta)\right]\\
&\sim\sum_{a=1}^{r}\left[\bar{\psi}_a(\mathrm{i}\gamma_\mu\partial_\mu+\mathrm{i}m_1)\psi_a-\frac{g}{2}j_{a\mu}^2+j_{a\mu}(\tilde{\partial}_\mu\theta)\right],
\end{aligned}
\tag{5.4}
$$

one can integrate over the disorder in h_1 ($\mathrm{i}m_1$) or in $\tilde{\partial}_\mu\theta$. In particular, integration over non-Gaussian h_1 ($\mathrm{i}m_1$) generates r^q, $q\in\mathbf{N}$, interactions through terms such as

$$
\left[\sum_{a=1}^{r}(\cos\chi_a)(\mathbf{x})\right]^q\sim\left[\sum_{a=1}^{r}(\bar{\psi}_a\psi_a)(\mathbf{x})\right]^q
\tag{5.5}
$$

in the replicated Lagrangian. According to conventional RG arguments, the importance of these operators is measured by their scaling dimensions at a given critical point labeled by $1/K$ and g_A. One then verifies along the derivation of Eq. (3.18) that, of all r^q operators,

$$
\mathcal{O}_{1\cdots 1}:=\prod_{a=1}^{q}e^{\mathrm{i}\chi_a}\sim\prod_{a=1}^{q}\bar{\psi}_a\psi_a,\ \mathcal{O}_q:=e^{\mathrm{i}q\chi_1}\sim(\bar{\psi}_1\psi_1)^q
\tag{5.6}
$$

are the most and less relevant, respectively, on the critical plane $m_1=0$, $1/K\geq 0$, $g_A\geq 0$:

$$
\overline{\langle\mathcal{O}^\dagger_{1\cdots 1}(\mathbf{x})\ \mathcal{O}_{1\cdots 1}(\mathbf{0})\rangle}\propto|\mathbf{x}|^{-2\pi Kq(1-g_AKq)},
\tag{5.7}
$$

$$
\overline{\langle\mathcal{O}^\dagger_{q}(\mathbf{x})\ \mathcal{O}_{q}(\mathbf{0})\rangle}\propto|\mathbf{x}|^{-2\pi Kq^2(1-g_AK)}.
\tag{5.8}
$$

The crucial point we want to make is that for any value of g_A, $\mathcal{O}_{1\cdots 1}$ becomes relevant as q is increased. Hence, conventional RG arguments predict that all critical points on the plane $1/K\geq 0$, $g_A>0$ are unstable with respect to random fugacity Y_1 (magnetic field h_1).

Next, we show that the existence of infinitely many relevant operators is not an artifact of an ill-defined replica limit. Indeed, there exists a very interesting line of critical points at twice the bare KT transition temperature $1/K=\pi$, $g_A>0$, $Y_1=0$ in the phase diagram of Fig. 1. From the perspective of the Thirring model, it describes massless non-interacting Dirac fermions in the presence of a transverse random vector potential. All disorder averaged correlation functions of local operators were calculated with the help of supersymmetric techniques appropriate to non-interacting systems [7,8]. It was found in [7,8] that each critical point has infinitely many primary fields and that these primary fields control the multifractal scaling of the random "wave function" $\exp[\theta(\mathbf{x})]$. These primary fields are precisely given by the family

$\mathcal{O}_{1\cdots 1}$ in the replica language and carry the scaling dimension $2q[1-(g_A/\pi)q]$ appropriate to $1/K=\pi$.

If we accept the hypothesis that having infinitely (as opposed to finitely) many relevant perturbations results in the instability of the line $1/K=\pi$, $g_A>0$, we need not even rely on the replica approach to rule out a KT transition. Indeed, the shape of the phase digram of Fig. 1 is preserved if we forbid thermal excitations of vorticity one ($Y_1=0$) but allow thermal excitations of vorticity two ($Y_2\geq 0$). More precisely, if we replace in the SG (Th) Lagrangian $\frac{h_1}{t}\cos\chi$ $(im_1\bar\psi\psi)$ by $\frac{h_2}{t}\cos(2\chi)$ $(im_2(\bar\psi\psi)^2)$, then the parabola $g_A(\frac{1}{K})=\frac{1}{K}(1-\frac{2}{\pi}\frac{1}{K})$ is simply rescaled to $\frac{1}{K}(1-\frac{1}{2^2}\times\frac{2}{\pi}\frac{1}{K})$. The inescapable conclusion is then that random fugacity for charge two vortices destroys this quasi-long-range ordered phase since it generates relevant interactions given by Eq. (5.7) under length rescaling.

VI. THE VILLAIN APPROXIMATION

In this section, we go back to the lattice to study in more details the nature of the Gaussian approximation made in Eq. (1.2). We consider both the random bond XY and Villain models on the square lattice. We will see that the difficulties with the fugacity expansion are not associated with pathologies of the field theory at short distances (such as ill-defined operator product expansions) but are intrinsic to the fugacity expansion, i.e., are also present if the fugacity expansion is performed on the Villain model itself. We will also see that randomness in the phase only ($g_A>0$, $g_J=0$) has different consequences in the XY and Villain models.

We begin with the random partition function

$$\mathcal{Z}_{XY}:=\int_0^{2\pi}\left(\prod_{i=1}^{L^2}\frac{d\phi_i}{2\pi}\right)\prod_{\langle ij\rangle}^{2L^2}e^{-\mathcal{L}^{XY}_{ij}}, \tag{6.1a}$$

$$\mathcal{L}^{XY}_{ij}:=K_{ij}\left[1-\cos(\phi_i-\phi_j-A_{ij})\right], \tag{6.1b}$$

for the random bond XY model on a square lattice made of L^2 sites. Directed links (two per site) on the square lattice are denoted by $\langle ij\rangle$. The phases $A_{ij}=-A_{ji}$ are random (with short-range correlations for different links). However, they need not be restricted to $0\leq A_{ij}<2\pi$ in spite of the periodicity of the cosine. Indeed, the probability distribution for A_{ij} need not be periodic with period 2π. The reduced spin stiffness $J_{ij}>0$ are also random (with short-range correlations for different links). A reasonable choice for the probability distribution of the random phases in the XY model is

$$P[A_{ij}]:=\prod_{\langle ij\rangle}^{2L^2}\frac{1}{\sqrt{2\pi g_A}}\,e^{-\frac{1}{2g_A}A^2_{ij}}. \tag{6.2}$$

This choice is made without loss of generality provided any "small" departure from Eq. (6.2) does not prevent the system to flow to the fixed point it would have reached otherwise. Although the random phase can take all possible real values, the energy per link $\mathcal{L}^{XY}_{ij}$ cannot take values outside the range (assumed compact for any *finite* temperature)

$$0\leq\mathcal{L}^{XY}_{ij}\leq 2\sup_{\langle ij\rangle}\ K_{ij} \tag{6.3}$$

with probability one.

The random Villain model consists in defining on the same lattice the random partition function [24–27]

$$\mathcal{Z}_{\rm V}:=e^{-KL^2}\int_0^{2\pi}\left(\prod_{i=1}^{L^2}\frac{d\phi_i}{2\pi}\right)\prod_{\langle ij\rangle}^{2L^2}\sum_{l_{ij}\in\mathbf{Z}}e^{-\mathcal{Q}^{\rm V}_{ij}},$$

$$\mathcal{Q}^{\rm V}_{ij}:=\frac{K_{ij}}{2}\left(\phi_i-\phi_j-A_{ij}-2\pi l_{ij}\right)^2. \tag{6.4}$$

Here, we have taken the spin stiffness to be selfaveraging, i.e.,

$$K:=\frac{1}{2L^2}\sum_{\langle ij\rangle}^{2L^2}K_{ij}=\frac{J}{T}+\mathcal{O}\left(\frac{1}{L}\right). \tag{6.5}$$

The periodicity under a shift of any ϕ_i by 2π is preserved in the Villain action, but the non-linearity of the cosine has been removed in the Villain action. When referring to the random Villain model, we will have in mind the *same* probability distribution for the spin stiffness and for the random phase as for the XY model.

The quantity $\mathcal{Q}^V_{ij}$ is not the counterpart to the link energy (6.1b) in the XY model since it is not periodic under a shift of A_{ij} or ϕ_i by an integer multiple of 2π. It is, however, very closely related to the energy of a given configuration of vortices in the background of a random environment induced by bond randomness.

By taking the random phase A_{ij} of the Villain model to be Gaussian distributed according to Eq. (6.2), we immediately see that $\mathcal{Q}^{\rm V}_{ij}$ can take any arbitrary large value with a finite probability. Moreover, $\exp\left(\mathcal{Q}^{\rm V}_{ij}\right)$ is log-normal distributed very much in the same way as correlation functions for vortex operators are in the Gaussian approximation (see section III and appendix A). This behavior should be contrasted with that of the Villain link energy

$$\mathcal{L}^{\rm V}_{ij}:=-\ln\left(\sum_{l_{ij}\in\mathbf{Z}}e^{-\mathcal{Q}^{\rm V}_{ij}}\right), \tag{6.6}$$

which is indeed periodic under a change of A_{ij} or ϕ_i by an integer multiple of 2π. It is crucial to realize that periodicity of the Villain link energy $\mathcal{L}^{\rm V}_{ij}$ is broken to any finite order in a fugacity expansion, since the fugacity expansion amounts to a truncation of the summation over $l_{ij}\in\mathbf{Z}$.

There is a noteworthy difference between the random bond XY and Villain models. If we take the spin stiffness

to be non-random but allow the relative phase between neighboring spins to be random, then we must assume that the SG model has both a random vector potential and a random fugacity if it is interpreted as the minimal model capturing the long distance properties of the XY model. To the contrary, the fugacity of the SG model is not random if it is derived from the Villain model with random phase but no randomness in the spin stiffness.

To clarify these last two points, we need first to introduce some notation. We define the longitudinal and transversal components $A^{\parallel}_{ij}$ and $A^{\perp}_{ij}$, respectively, by

$$A_{ij} := A^{\parallel}_{ij} + A^{\perp}_{ij}, \tag{6.7}$$

where $A^{\parallel}_{ij}$ is curl free, i.e.,

$$\begin{aligned} 0 =& \ \mathrm{curl}_{\mathbf{i}}\ A^{\parallel}_{ij} \\ :=& A^{\parallel}_{i(i+\hat{\mathbf{x}})} + A^{\parallel}_{(i+\hat{\mathbf{x}})(i+\mathbf{x}+\hat{\mathbf{y}})} + A^{\parallel}_{(i+\hat{\mathbf{x}}+\mathbf{y})(i+\hat{\mathbf{y}})} + A^{\parallel}_{(i+\hat{\mathbf{y}})i}, \end{aligned} \tag{6.8}$$

and $A^{\perp}_{ij}$ is divergence free, i.e.,

$$\begin{aligned} 0 =& \ \mathrm{div}_{i}\ A^{\perp}_{ij} \\ :=& A^{\perp}_{i(i+\hat{\mathbf{x}})} - A^{\perp}_{(i-\hat{\mathbf{x}})i} + A^{\perp}_{i(i+\hat{\mathbf{y}})} - A^{\perp}_{(i-\hat{\mathbf{y}})i}. \end{aligned} \tag{6.9}$$

Dual sites are labeled by

$$\mathbf{i} := i + \frac{1}{2}\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{y}}, \tag{6.10}$$

where the basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ span the square lattice. For completeness, the gradient of a lattice scalar is defined by

$$\mathrm{grad}_{\hat{\mu}}\phi_i := \phi_{i+\hat{\mu}} - \phi_i, \qquad \hat{\mu} = \hat{\mathbf{x}}, \hat{\mathbf{y}}. \tag{6.11}$$

It is possible to rewrite the Villain partition function solely in terms of degrees of freedom defined on the dual lattice [24–27],

$$\mathcal{Z}_{\mathrm{V}} \propto \sum_{\{l^{\perp}_{\{\mathbf{ij}\}}\} \in \mathbf{Z}^{2L^2}} \int\limits_{-\infty}^{+\infty} \left(\prod_{\mathbf{i}=1}^{L^2} \frac{d\varphi_{\mathbf{i}}}{2\pi} \right) e^{-\frac{1}{2}\sum\limits_{\langle \mathbf{ij}\rangle}^{2L^2} \left[\mathcal{Q}^{\mathrm{SW}}_{\mathbf{i}(\mathbf{i}+\hat{\mu})} + \mathcal{Q}^{\mathrm{CB}}_{\mathbf{i}(\mathbf{i}+\hat{\mu})}\right]}. \tag{6.12}$$

We have dropped a multiplicative factor that depends on the (random) spin stiffness, and

$$\begin{aligned} &\mathcal{Q}^{\mathrm{SW}}_{\mathbf{i}(\mathbf{i}+\hat{\mu})} + \mathcal{Q}^{\mathrm{CB}}_{\mathbf{i}(\mathbf{i}+\hat{\mu})} := \\ &\frac{\left(\mathrm{grad}_{\hat{\mu}}\varphi_{\mathbf{i}}\right)^2}{K_{\mathbf{i}(\mathbf{i}+\hat{\mu})}} + 2\mathrm{i}\left(\mathrm{grad}_{\hat{\mu}}\varphi_{\mathbf{i}}\right)\left(A^{\perp}_{\mathbf{i}(\mathbf{i}+\hat{\mu})} + 2\pi l^{\perp}_{\mathbf{i}(\mathbf{i}+\hat{\mu})}\right). \end{aligned} \tag{6.13}$$

The random dual phases $A^{\perp}_{\mathbf{ij}}$ are now curl free as are dual vortex degrees of freedom $l^{\perp}_{\mathbf{ij}}$.

Since Eq. (6.13) is quadratic in the spin wave degrees of freedom $\varphi_{\mathbf{i}}$ whereas Eq. (6.13) is linear in the vortex degrees of freedom $l^{\perp}_{\mathbf{ij}}$, it is possible to decouple the spin wave sector from the vortex sector. Our final expression for the partition function in the vortex sector is [compare with Eq. (3.9)]

$$\mathcal{Z}_{\mathrm{CB}} = \sum_{\{m_{\mathbf{i}}\}} \prod_{\mathbf{i},\mathbf{j}=1}^{L^2} e^{-\pi(m_{\mathbf{i}}-n_{\mathbf{i}})(m_{\mathbf{j}}-n_{\mathbf{j}})D_{\mathbf{ij}}[K_{\mathbf{kl}}]}, \tag{6.14}$$

$$\mathrm{div}_{\mathbf{i}} l^{\perp}_{\mathbf{ij}} =: \frac{1}{\sqrt{2\pi}} m_{\mathbf{i}}, \tag{6.15}$$

$$\mathrm{div}_{\mathbf{i}} A^{\perp}_{\mathbf{ij}} =: \sqrt{2\pi} n_{\mathbf{i}}. \tag{6.16}$$

Here, $D_{\mathbf{ij}}[K_{\mathbf{kl}}]$ are the random components of the dual lattice Green function in the background of random spin stiffness, namely the inverse of the quadratic form in the spin wave sector.

Performing a dual transformation on the random bond Villain model thus offers two insights. First, given a Gaussian probability distribution for the random phases and no randomness in the spin stiffness, the probability distribution for the CB energy of a given configuration of vortices $\{m_{\mathbf{i}}\}$ is Gaussian, and exponentiating this CB energy yields a log-normal distributed random variable. We thus conclude that *the existence of log-normal distributed correlation functions in the random spin wave phase is not an artifact of the continuum limit.*

Second, the diagonal components $D_{\mathbf{ii}}[K_{\mathbf{kl}}]$ of the Green function define the vortex core energy. *Hence, in the absence of random spin stiffness, the core energy of the vortex sector is not random within the Villain model.*

We recover the XY partition function by replacing the right hand side of Eq. (6.4) by

$$-K_{ij}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left(\phi_i - \phi_j - A_{ij} - 2\pi l_{ij}\right)^{2n}. \tag{6.17}$$

Spin waves and vortices are coupled in the XY model with or without randomness. In an effective theory such as the SG model, non-linearities such as in Eq. (6.17) can be incorporated through a random fugacity to a first approximation. In the clean limit, the irrelevance of higher vorticity charges justifies neglecting such an effect. However, as we have shown in section V this is not anymore the case in the presence of random phases.

A random core energy for vortices is always induced by a random spin stiffness. However, we have shown that the XY and Villain models differ in one very important aspect when only the relative phase between neighboring spins is random. Indeed, due to non-linear effects the core energy of vortices is always random in the XY model with random phase whereas this is not the case for the Villain model. If randomness in the core energy does indeed destroy the exotic quasi-long-range order proposed in [20], we must then conclude that the Villain model with random phase only does not belong to the same universality class as the XY model with random phase only *in a strong sense.* It would be very interesting to

probe numerically our conjectured difference between the random XY and Villain models.

We have also shown that the random XY model and any approximative treatment of the Villain model based on a perturbative expansion in the fugacity do not belong to the same universality class in *a weak sense*. We can illustrate this fact in a very suggestive way. We take the spin-spin correlation function (thermal average) in the random bond XY model for an arbitrary pair of sites i and j. Clearly, this is a random variable that takes values on a compact range, namely

$$|\langle \cos\phi_i \cos\phi_j \rangle| \leq 1. \tag{6.18}$$

Consequently, all integer moments with $p < q$ obey

$$\overline{|\langle \cos\phi_i \cos\phi_j \rangle|^p} \leq 1, \tag{6.19}$$

$$\overline{|\langle \cos\phi_i \cos\phi_j \rangle|^q} \leq \overline{|\langle \cos\phi_i \cos\phi_j \rangle|^p} \leq 1. \tag{6.20}$$

It is possible to verify that those inequalities are not satisfied to any finite order in a fugacity expansion of the random phase Villain model. Without loss of generality, we can take the continuum limit. We recall that the prescription to estimate the spin-spin correlation function (thermal average) from the continuum theory of section III is to identify [25]

$$\langle \cos\phi_i \cos\phi_j \rangle_{\mathcal{Z}_{XY}} \to \left\langle e^{-\frac{1}{2\pi K}\int_{\mathbf{x}_j}^{\mathbf{x}_i} ds_\mu \tilde{\partial}_\mu \chi} \right\rangle_{\mathcal{Z}_{\mathrm{SG}}} \tag{6.21}$$

for some path joining $\mathbf{x}_j$ to $\mathbf{x}_i$. To any finite order in a fugacity expansion, the spin-spin correlation function (thermal average) is very broadly distributed (log-normal in the limit of infinite core energy), and thus violates the bounds of Eqs. (6.19,6.20). We stress that this is a failure of the fugacity expansion on the Villain model itself and not an artifact of the continuum limit [2].

This situation is very reminiscent of that in the one-dimensional random bond Ising model. For a large but fixed separation, the Ising spin-spin correlation function (thermal average) is "close" to being log-normal [2]. The log-normal approximation describes exactly the mean and variance of the logarithm of the spin-spin correlation function (thermal average) in the Ising case but it neglects higher cumulants. By analogy, we might expect that the fugacity expansion on the Villain model captures well the logarithm of the XY spin-spin correlation (thermal average). However, the log-normal approximation in the random bond Ising model fails badly to describe the tails of the Ising spin-spin correlation functions very much in the same way as the fugacity expansion on the Villain model dramatically overestimate tails for the probability distribution of the XY spin-spin correlation functions.

So it is by now clear that the probability distribution of the random vector potential unduly favors rare events through its tails within the fugacity expansion. In one scenario, we must then anticipate partial loss of universality for the fixed point probability distribution of correlation functions (thermal average) that are broadly distributed, since their tails result from rare events. We are aware of several examples of this kind: directed polymers in a random medium [2], the metal-insulator transition [29–31], and quantum gravity [10]. In the worst case scenario, the fugacity expansion on the Villain model is overwhelmed by rare events and loses complete predictability with regard to the phase diagram of the XY model.

VII. CONCLUSIONS

Concerned with the possibility that effective theories describing random critical points are often characterized by a spectrum of scaling exponents that is unbounded from below, we have studied the Gaussian approximation to the random XY model on a square lattice within a perturbative RG framework.

The Gaussian approximation to the random XY model predicts the existence of a manifold of random critical points describing a random spin wave phase. However, there are correlation functions in the random spin wave phase that are log-normal distributed. Correspondingly, there are infinitely many operators with negative scaling dimensions that are associated with vortices in the effective theory describing the random spin wave phase.

We showed that all these operators with negative scaling dimensions contribute in a highly non-trivial way to the perturbative RG equations within a fugacity expansion around the random spin wave phase. The existence of infinitely many negative scaling dimensions thus manifests itself by the non-analyticity of the fugacity expansion for any given temperature and disorder strength.

[2] An alternative way to stress this point is to note that the exact identity between the two- and four-points spin-spin correlation functions [28]

$$\overline{\langle e^{\mathrm{i}(\phi_i-\phi_j)}\rangle} = \overline{|\langle e^{\mathrm{i}(\phi_i-\phi_j)}\rangle|^2} \tag{6.22}$$

that holds along the Nishimori line $K = 1/g_A$ in the Villain model is always violated to any finite order in the fugacity expansion.

In this sense, the random spin wave phase is unstable, although we cannot preclude the possibility that a new phase with quasi-long-range order can be found for sufficiently low temperatures and weak disorder strength. The breakdown of the fugacity expansion is also associated to a potential perturbative instability triggered by any random vortex core energy.

We have shown that neither the breakdown of the fugacity expansion nor the perturbative instability are artifacts of the continuum approximation that we used but would also be present in the random bond Villain model on the square lattice. Rather, they both reflect the extreme sensitivity of the fugacity expansion to the tails of the probability distribution that is chosen for the random bonds.

The physical interpretation for the breakdown of the fugacity expansion is that the typical ground state is not ferromagnetic. In the best case scenario, the typical ground state for weak disorder supports some quasi-long-range order that would persist for sufficiently low temperatures. However, to address the nature of the low temperature, weak disorder region of the phase diagram it is imperative to better characterize the typical ground state and to use a RG scheme that is non-perturbative in the vortex fugacity.

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APPENDIX A: ZERO TEMPERATURE CONSIDERATIONS

In this appendix, we study some properties of the random phase XY model at zero temperature. Indeed, we recall that the prerequisite to the existence of a KT transition in the disorder free XY model is the simple fact that the ground state of the Hamiltonian

$$H[\{\phi_i\}] := \sum_{\langle ij\rangle} J[1-\cos(\phi_i-\phi_j)] \tag{A1}$$

is ferromagnetic. In two dimensions, the long-range order of the ground state is first demoted to quasi-long-range order for any finite temperature below the KT transition temperature T_{KT}. In turn, topological excitations in the form of vortices wipe out the quasi-long-range order for $T > T_{\mathrm{KT}}$. The crucial question we would like to address is: are typical ground states of

$$H[\{\phi_i\};\{A_{ij}\}] := \sum_{\langle ij\rangle} J[1-\cos(\phi_i-\phi_j-A_{ij})] \tag{A2}$$

ordered, quasi-long-range ordered, or disordered?

What we mean by a typical ground state is the following. We assume that we know how to calculate the probability to find a ground state with energy density e (energy divided by total number of sites, i.e., $0 \le e \le 4J$). A typical ground state maximalizes this probability distribution. Unfortunately, obtaining this probability distribution is very difficult in view of: (i) the non-quadratic dependency of the energy on the disorder, and (ii) of the need to minimalize the energy spectrum of $H[\{\phi_i\};\{A_{ij}\}]$ for a given realization of the disorder. The first difficulty can be disposed of in the Villain approximation (whereby it is assumed that the Villain approximation does not change the universality class), but we still must face the second difficulty.

We focus on the vortex sector in the continuum approximation of section III and show that, for extensively many realizations of the random vector potential $\tilde{\partial}_\mu\theta$, the ground states are, for lack of a better description, *complex* configurations of vortices. We do this by generalizing an argument used in [20] to prove that the random potential $\tilde{\partial}_\mu\theta$ destroys the long-range order of the pure system at $T=0$ and for sufficiently strong disorder strength: $g_A > \pi/8$. We then modify the random phase XY model by restricting the possible realizations of the disorder to a more tractable subset. Within this subset we show that it is possible to decide whether the ground states support quasi-long-range order or not.

1. Gaussian distribution of the energy of vortex configurations

Let $\tilde{\Theta}$ in Eq. (3.3) be given by the vortex configuration

$$\tilde{\Theta}(\mathbf{x}) := \sum_{i=1}^{n} m_i \ln\left|\frac{\mathbf{x}-\mathbf{x}_i}{l_0}\right|, \quad m_i \in \mathbf{Z}, \tag{A3}$$

(which need not be neutral) and define its energy in the background of the random vector potential $\tilde{\partial}_\mu\theta$ to be

$$\begin{aligned} H_{1,\cdots,n} &:= \frac{J}{2}\int d^2\mathbf{x}\ \left[(\tilde{\partial}_\mu\tilde{\Theta})^2 - 2(\tilde{\partial}_\mu\tilde{\Theta})(\tilde{\partial}_\mu\theta)\right] \\ &\equiv \overline{H_{1,\cdots,n}} + \delta H_{1,\cdots,n}, \end{aligned} \tag{A4}$$

where

$$\overline{H_{1,\cdots,n}} = -\pi J \sum_{k,l=1}^{n} m_k m_l \ln \left| \frac{\mathbf{x}_k - \mathbf{x}_l}{l_0} \right|,$$

$$\delta H_{1,\cdots,n} = 2\pi J \sum_{k=1}^{n} m_k \theta(\mathbf{x}_k). \tag{A5}$$

The definition in Eq. (A4) is useful since the energy $H_{1,\cdots,n}$ is very closely related to the thermal average of the $2m = n$-point correlation function

$$F_{\mathbf{x}_1,\cdots,\mathbf{x}_{2m}} := e^{\mathrm{i}\chi(\mathbf{x}_1)} \cdots e^{\mathrm{i}\chi(\mathbf{x}_m)} e^{-\mathrm{i}\chi(\mathbf{x}_{m+1})} \cdots e^{-\mathrm{i}\chi(\mathbf{x}_{2m})}, \tag{A6}$$

for vanishing fugacity $h_1/2t$. Indeed, in that case

$$\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_{2m}} \rangle_{h_1=0} = e^{-\frac{K}{J} H_{1,\cdots,2m}}. \tag{A7}$$

Notice that for a *fixed* realization of $\tilde{\partial}_\mu \theta$, $H_{1,\cdots,n}$ is *bounded from below* by $-\frac{J}{2} \int d^2\mathbf{x} (\tilde{\partial}_\mu \theta)^2$. On the other hand, since $\frac{J}{2} \int d^2\mathbf{x} (\tilde{\partial}_\mu \theta)^2$ can take arbitrary large values, these are the rare events which take full advantage of the non-compactness of the Gaussian probability distribution for $\tilde{\partial}_\mu \theta$, the probability distribution $P(E; \mathbf{x}_1, \cdots, \mathbf{x}_n)$ that $H_{1,\cdots,n}$ takes the value E is non-vanishing for all real values of E. A very special vortex configuration is the *ferromagnetic* configuration $\tilde{\Theta} = 0$. This is the ground state of the pure system and it has a vanishing random energy. Clearly it need not be the ground state for a given realization of the disorder.

What is the probability distribution of $H_{1,\cdots,n}$? By definition it is given by

$$P(E; \mathbf{x}_1, \cdots, \mathbf{x}_n) := \frac{\int \mathcal{D}[\theta]\ e^{-\frac{1}{2g_A} \int d^2\mathbf{y}\ (\partial_\mu \theta)^2(\mathbf{y})}\ \delta(E - H_{1,\cdots,n})}{\int \mathcal{D}[\theta]\ e^{-\frac{1}{2g_A} \int d^2\mathbf{z}\ (\partial_\mu \theta)^2(\mathbf{z})}}. \tag{A8}$$

We can represent the delta function by an integral, in which case

$$\begin{aligned} P(E; \mathbf{x}_1, \cdots, \mathbf{x}_n) &= \int \frac{d\lambda}{2\pi}\ e^{\mathrm{i}\lambda\left(E - \overline{H_{1,\cdots,n}}\right)}\ \overline{e^{-\mathrm{i}\lambda \delta H_{1,\cdots,n}}} \\ &= \int \frac{d\lambda}{2\pi}\ e^{\mathrm{i}\lambda\left(E - \overline{H_{1,\cdots,n}}\right)}\ e^{-\frac{\lambda^2}{2} \overline{(\delta H_{1,\cdots,n})^2}} \\ &= \frac{\exp\left[-\frac{\left(E - \overline{H_{1,\cdots,n}}\right)^2}{2\overline{(\delta H_{1,\cdots,n})^2}}\right]}{\sqrt{2\pi \overline{(\delta H_{1,\cdots,n})^2}}}. \end{aligned} \tag{A9}$$

Thus, the probability distribution for the energy of the vortex configuration $\tilde{\Theta}(\mathbf{x})$ is Gaussian with mean $\overline{H_{1,\cdots,n}}$ and variance

$$\begin{aligned} \overline{(\delta H_{1,\cdots,n})^2} &= -2\pi J^2 g_A \sum_{k,l=1}^{n} m_k m_l \ln \left| \frac{\mathbf{x}_k - \mathbf{x}_l}{l_0} \right| \\ &= 2 J g_A\ \overline{H_{1,\cdots,n}}. \end{aligned} \tag{A10}$$

We are now ready to calculate the (unnormalized) distribution $P(E; m_1, \cdots, m_n)$ to find n vortices with vorticities $m_1, \cdots, m_n \in \mathbf{Z}$ anywhere on the Euclidean plane. It is defined by

$$P(E; m_1, \cdots, m_n) := \int \frac{d^2\mathbf{x}_1}{a^2} \cdots \frac{d^2\mathbf{x}_n}{a^2} P(E; \mathbf{x}_1, \cdots, \mathbf{x}_n), \tag{A11}$$

where a is a microscopic cutoff and L is the system size. We are interested in the scaling of $P(E; m_1, \cdots, m_n)$ with the system size L. Notice that

$$\int dE\ P(E; m_1, \cdots, m_n) = \left(\frac{L}{a}\right)^{2n}. \tag{A12}$$

This scaling can be calculated in closed form for $n = 1$:

$$P(E; m) = \frac{\exp\left[\ln\left(\frac{L}{a}\right)^2 - \frac{\left(E - \overline{H_1}\right)^2}{4 J g_A \overline{H_1}}\right]}{\sqrt{4\pi J g_A \overline{H_1}}}, \tag{A13}$$

$$\overline{H_1} = m^2 \pi J \ln\left(\frac{L}{a}\right). \tag{A14}$$

Here, the arbitrary length scale l_0 is chosen to be L.

In the absence of disorder, the ground state of Eq. (A4) is the ferromagnetic state $\tilde{\Theta} = 0$. In the presence of disorder, we can calculate the probability $P(E; \mathbf{x}_1, \cdots, \mathbf{x}_n)$ from Eq. (A9) to find a vortex configuration with energy $E < 0$. We say that it is energetically favorable to create a vortex configuration $\tilde{\Theta}$ different from $\tilde{\Theta} = 0$ if $P(E < 0; \mathbf{x}_1, \cdots, \mathbf{x}_n)$ does not vanish. For example, to a good approximation, we find that

$$\begin{aligned} \int_{-\infty}^{0} dE\ P(E; m) &\approx\ P(0; m) \\ &= \frac{\left(\frac{L}{a}\right)^{2\left(1 - \frac{\pi m^2}{8 g_A}\right)}}{\sqrt{4\pi^2 J^2 g_A m^2 \ln(L/a)}}. \end{aligned} \tag{A15}$$

Equation (A15) gives the number of sites on which it is energetically favorable to create a single vortex configuration.

As proposed in [20], one can use estimate Eq. (A15) to establish a *criterion* for the destruction of long-range order at $T = 0$ of the pure system by the random vector potential $\tilde{\partial}_\mu \theta$. Long-range order is destroyed if the number of sites on which it is energetically favorable to create a single vortex configuration diverges in the thermodynamic limit $L \uparrow \infty$, i.e., if the variance g_A is larger than the critical value

$$(g_A)_{\text{crit}} := \frac{\pi}{8} m^2 = \left(\frac{m}{2}\right)^2 \times \frac{\pi}{2}. \tag{A16}$$

It is important to stress that this condition does not guaranty the existence of quasi-long-range order at $T = 0$ if $g_A < (g_A)_{\text{crit}}$. To see this one can estimate the

number of pairs of sites on which it is energetically favorable to create a dipole. This number is approximately $P(0,+m,-m)$. In turn, with the choice $l_0 = a$, $P(0,\mathbf{x}_1,\mathbf{x}_2)$ can be read off from $P(0,\mathbf{x}_1)$ provided one replaces $\ln(L/a)$ in $P(0,\mathbf{x}_1)$ by $2\ln(|\mathbf{x}_1-\mathbf{x}_2|/a)$:

$$P(0,+m,-m) = \frac{4\pi}{\sqrt{8\pi^2 J^2 g_A m^2}}\left(\frac{L}{a}\right)^2 \int_0^{\sqrt{\ln(L/a)}} dy\, e^{2\left(1-\frac{\pi m^2}{4 g_A}\right)y^2}. \quad \text{(A17)}$$

Hence, the number of pairs of sites on which it is energetically favorable to create a dipole *always* diverges with system size [like $(L/a)^2$ if $g_A < 2(g_A)_{\rm crit}$, like $(L/a)^{4-(\pi m^2)/(2g_A)}$ otherwise]. Likewise, the number of n sites on which it is energetically favorable to create a *neutral* vortex configuration always diverge with system size. Hence, for most realizations of the disorder $\tilde{\partial}_\mu\theta$, the ground states are not the ferromagnetic state $\tilde\Theta = 0$ but are neutral and non-trivial (*complex*) vortex configurations. To decide whether such complex ground states support quasi-long-range order, one must establish absence of screening of the CB potential between to external charges.

2. Screening at zero temperature

To illustrate the issue of screening, we introduce the random energy

$$H_{\rm CB}[\tilde\Theta,\theta_\alpha] := -\pi J \sum_{k,l}(m_k - n_k)G_{kl}(m_l - n_l), \quad \text{(A18)}$$

where $\tilde\Theta$ is a neutral configuration of charges $m_k = 0,\pm1$, θ_α is a neutral configuration of charges $n_k = 0,\pm\alpha$, $0<\alpha<1$, that realizes the disorder, and G_{kl} is a short hand notation for the logarithmic CB gas potential. Finally, we take the probability to realize θ_α to be proportional to

$$\exp\left[-\frac{\pi}{g_A}\sum_{k,l} n_k G_{kl} n_l\right]. \quad \text{(A19)}$$

The relationship between this model and the random phase XY model is that only a very small subset of all possible realizations of the disorder in the random phase XY model are allowed in Eqs. (A18,A19). Thrown out are all realizations of the disorder made of vortices of unequal vorticity.

Consider now the vortex configuration Ξ_α defined by $m_k = 0,\pm1$, respectively, whenever $n_k = 0,\pm\alpha$. The energy of this configuration, whose unit vortices track the fractionally charged quenched vortices, is

$$H_{\rm CB}[\Xi_\alpha,\theta_\alpha] := -\pi J(1-\alpha)^2\sum_{k,l} m_k G_{kl} m_l, \quad \text{(A20)}$$

and should be compared to the energy of the ferromagnetic state

$$H_{\rm CB}[0,\theta_\alpha] := -\pi J\alpha^2\sum_{k,l} m_k G_{kl} m_l. \quad \text{(A21)}$$

We see that it is energetically favorable to create a unit vortex m_k at the location of every quenched vortex n_k provided $1/2<\alpha<1$. Otherwise, the ferromagnetic state is preferred. For $\alpha = 1/2$, Ξ_α and θ_α are degenerate [3].

We are in position to ask the following question. Is the CB potential screened or not for the family of vortex configurations $\{\Xi_\alpha\}$ where $1/2<\alpha<1$ is held fixed? Since Ξ_α merely creates vortices wherever quenched vortices sit, the question can be reduced to: what are the screening properties of the CB gas with effective temperature and charge g_A and α, respectively? The answer is known [12], namely for sufficiently small g_A, i.e.,

$$g_A < \alpha^2\frac{\pi}{2}, \quad \text{(A22)}$$

the CB gas does not screen since the dipole phase is realized and there exists quasi-long-range order. Otherwise, the CB gas screens since the plasma phase is realized and quasi-long-range order is destroyed.

Once we know the asymptotic form of the CB potential between two external charges for the family $\{\Xi_\alpha\}$, $1/2<\alpha<1$ and g_A fixed, we can extract the α-averaged CB potential where we restrict $1/2<\alpha<1$. The α-averaged CB potential is controlled in the thermodynamic limit by $\alpha = 1/2$ since the probability in Eq. (A19) scales like $\exp[-\alpha^2(L/a)^2 f(L/a)]$, where $f(x)$ is some positive function with $\lim_{x\uparrow\infty} f(x)/x^2 = 0$ that does not depend on α. The α-averaged CB potential, $1/2<\alpha<1$, thus decays logarithmically for sufficiently small g_A and decays exponentially if $g_A > (g_A)_{\rm crit}$. Hence, quasi-long-range order is present for not too strong disorder strength provided it can be shown that Ξ_α is the ground state for every realizations of the disorder when $1/2<\alpha<1$.

Although this conjecture might be reasonable for the toy model, it is certainly not true for the full problem where a given realization of the disorder $\tilde\partial_\mu\theta$ can create vortices of arbitrary fractional charges in contrast to the toy model. Hence, although the existence of both θ_α and Ξ_α suggests the existence of $(g_A)_{\rm crit}$ in the full model, we cannot rule out more complex configurations $\tilde\Theta$ with energies lower than that of both the ferromagnetic state and Ξ_α and for which the CB potentials are screened at long distances for some $g_A < (g_A)_{\rm crit}$.

[3] We are indebted to E. Fradkin for this observation.

APPENDIX B: FUGACITY EXPANSION

In this section, we we are going to expand all correlation functions of the SG operator $e^{\mathrm{i}\chi}$ in powers of $h_1/2t$ and then perform disorder averaging order by order in powers of $h_1/2t$. We thus reproduce the fugacity expansion performed by Korshunov [21] on the CB gas with quenched fractionally charged vortices. Scaling fields with negative scaling dimensions are then seen to control the singular behavior of the expansion in powers of $h_1/2t$ of the inverse SG partition function order by order. However, we begin first by illustrating a possible drawback of the fugacity expansion.

1. Drawback of the fugacity expansion

The expansion of the inverse SG partition function in powers of the fugacity $h_1/2t$ should be taken with great caution since convergence is not always warranted. Indeed, let X be a real valued random variable and let $Y := 1/(1+X)$ be another real valued random variable. For example, Y could be $1/\mathcal{Z}_{\mathrm{CB}}$ whereby X could be $\mathcal{Z}_{\mathrm{CB}} - 1$. Let $P_X(x)$ be the probability that X takes the value x. We want to calculate the probability $P_Y(y)$ that Y takes the value y. For definiteness,

- Case I:

$$\begin{aligned} P_X(x) :=&\ e^{-x}, \quad 0 \le x < \infty \Leftrightarrow \\ P_Y(y) =&\ \frac{e^{1-\frac{1}{y}}}{y^2}, \quad 0 \le y \le 1. \end{aligned} \tag{B1}$$

- Case II:

$$\begin{aligned} P_X(x) :=&\ x^{-2}, \quad 1 \le x < \infty \Leftrightarrow \\ P_Y(y) =&\ (1-y)^{-2}, \quad 0 \le y \le \frac{1}{2}. \end{aligned} \tag{B2}$$

We immediately conclude that the expansion

$$Y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} X^n, \tag{B3}$$

would predict that all moments of Y diverge in both cases I and II. However, these moments can be calculated directly from Eqs. (B1,B2) and are all finite.

2. Preview to the fugacity expansion to fourth order

Let n be a positive integer, choose n points $\mathbf{x}_1, \cdots, \mathbf{x}_n$ on the Euclidean plane, and define

$$F_{\mathbf{x}_1,\cdots,\mathbf{x}_n} := e^{\mathrm{i}\varepsilon_1\chi(\mathbf{x}_1)} \cdots e^{\mathrm{i}\varepsilon_n\chi(\mathbf{x}_n)}, \tag{B4}$$

$$\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}} := \int \mathcal{D}[\chi]\ e^{-S_{\mathrm{SG}}[\chi,\theta]} F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}. \tag{B5}$$

Each factor $e^{\mathrm{i}\varepsilon_k\chi(\mathbf{x}_k)}$, $k = 1, \cdots, n$, can be thought of as the insertion of an external vortex of vorticity $\varepsilon_k = \pm 1$ in the CB gas. Thermal averaging of Eq. (B4) is obtained by dividing the unnormalized average in Eq. (B5) by the SG partition function:

$$\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle = \frac{\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}}{\mathcal{Z}_{\mathrm{SG}}[\theta]}. \tag{B6}$$

Finally, disorder averaging over $\tilde{\partial}_\mu\theta$ is done with the probability distribution of Eq. (3.4).

We attempt to calculate both $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle$ and $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}$ through a power expansion in $h_1/2t$ and then perform term by term disorder averaging over $\tilde{\partial}_\mu\theta$. Thermal averaging and $\tilde{\partial}_\mu\theta$ averaging are seen to "factorize" to each order in $h_1/2t$ for $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}$. We then go on performing $\tilde{\partial}_\mu\theta$ averaging over $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle$.

The key identity that is needed is

$$\begin{aligned} &\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}} = \\ &\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{h_1}{2t}\right)^m \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} \int d^2\mathbf{y}_1 \cdots \int d^2\mathbf{y}_m \times \\ &\left\langle e^{\mathrm{i}\chi(\mathbf{y}_1)} \cdots e^{\mathrm{i}\chi(\mathbf{y}_p)} e^{-\mathrm{i}\chi(\mathbf{y}_{p+1})} \cdots e^{-\mathrm{i}\chi(\mathbf{y}_m)} F_{\mathbf{x}_1,\cdots,\mathbf{x}_n} \right\rangle^{\mathrm{unnor}}_{h_1=0}. \end{aligned} \tag{B7}$$

Thermal averaging on the last line must be performed with $h_1 = 0$, in which case averaging over χ is Gaussian. We can then use the shift of integration variable $\chi =: \chi' + \frac{\mathrm{i}t}{2\pi}\theta$ to decouple averaging over χ' from averaging over $\tilde{\partial}_\mu\theta$:

$$\begin{aligned} &\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}} = \\ &\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{h_1}{2t}\right)^m \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} \int d^2\mathbf{y}_1 \cdots \int d^2\mathbf{y}_m \times \\ &e^{-\frac{t}{2\pi}\theta(\mathbf{y}_1)} \cdots e^{\frac{t}{2\pi}\theta(\mathbf{y}_m)} e^{-\frac{t}{2\pi}\sum_{k=1}^{n}\varepsilon_k\theta(\mathbf{x}_k)} \times \\ &\left\langle e^{\mathrm{i}\chi'(\mathbf{y}_1)} \cdots e^{-\mathrm{i}\chi'(\mathbf{y}_m)} e^{\mathrm{i}\sum_{k=1}^{n}\varepsilon_k\chi'(\mathbf{x}_k)} \right\rangle^{\mathrm{unnor}}_{h_1=0}. \end{aligned} \tag{B8}$$

Thus to each order in $h_1/2t$, averaging over $\tilde{\partial}_\mu\theta$ has factorized from averaging over χ' in the integrand on the right hand side of Eq. (B8). In fact since both averages are Gaussian, one obtains the CB gas representation

$$\begin{aligned} &\overline{\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}} = \\ &\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{h_1}{2t}\right)^m \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} \int d^2\mathbf{y}_1 \cdots \int d^2\mathbf{y}_m \times \\ &\exp\left[+\pi\bar{K} \sum_{k,l=1}^{m+n} \varepsilon_k\varepsilon_l \ln\left|\frac{\mathbf{z}_k - \mathbf{z}_l}{l_0}\right|\right], \end{aligned} \tag{B9}$$

where

$$\mathbf{z}_k = \begin{cases} \mathbf{y}_k, & \text{if } k=1,\cdots,m, \\ \mathbf{x}_k, & \text{if } k=m+1,\cdots,m+n, \end{cases} \tag{B10}$$

and we have introduced the effective CB gas coupling constant

$$2\pi\bar{K} := 2\pi(K-g_A K^2) = 2\pi\left[\frac{t}{4\pi^2}+\frac{g_A}{4\pi^2}\times\left(\mathrm{i}\frac{t}{2\pi}\right)^2\right]. \tag{B11}$$

Equations (B9,B11) tell us that under the assumption that a fugacity expansion of the CB gas is valid for each realizations of the disorder $\tilde{\partial}_\mu\theta$, then averaging over $\tilde{\partial}_\mu\theta$ all unnormalized correlation functions consisting in the insertion of external charges amounts to the *same* renormalization of the CB gas effective temperature $1/\pi K$. Moreover, in view of

$$\mathcal{Z}_{\mathrm{SG}}[\theta] = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{h_1}{2t}\right)^m\sum_{p=0}^{m}\binom{m}{p}\int d^2\mathbf{y}_1\cdots d^2\mathbf{y}_m \times \left\langle e^{\mathrm{i}\chi(\mathbf{y}_1)}\cdots e^{\mathrm{i}\chi(\mathbf{y}_p)}e^{-\mathrm{i}\chi(\mathbf{y}_{p+1})}\cdots e^{-\mathrm{i}\chi(\mathbf{y}_m)}\right\rangle^{\mathrm{unnor}}_{h_1=0}, \tag{B12}$$

this is also true for the case $n=0$ which corresponds to the average of the partition function expanded in powers of the fugacity. We thus expect that the fugacity expansion for all unnormalized correlation functions $\overline{\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}}$, if well defined, is convergent in the same region of the $1/K$, g_A, $h_1/2t$ coupling space. However, this is not so for the normalized average $\overline{\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle}$.

Indeed Eqs. (B9,B11) are not sufficient to extract the effect of disorder averaging over $\tilde{\partial}_\mu\theta$ on the perturbative expansion in powers of $h_1/2t$ of $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle$. In fact, we need *all moments* (not only the first one) of the unnormalized correlation functions $\langle F_{\mathbf{x}_1,\cdots,\mathbf{x}_n}\rangle^{\mathrm{unnor}}$ as can be seen by expanding $1/\mathcal{Z}_{\mathrm{SG}}[\theta]$ in powers of $h_1/2t$ in Eq. (B6). But these moments can be estimated from

$$\overline{\left[\left\langle e^{\mathrm{i}\chi(\mathbf{y}_1)}\cdots e^{\mathrm{i}\chi(\mathbf{y}_p)}e^{-\mathrm{i}\chi(\mathbf{y}_{p+1})}\cdots e^{-\mathrm{i}\chi(\mathbf{y}_m)}\right\rangle^{\mathrm{unnor}}_{h_1=0}\right]^q} = \exp\left[+\pi\overline{K(q)}\sum_{k,l=1}^{m}\varepsilon_k\varepsilon_l\ln\left|\frac{\mathbf{y}_k-\mathbf{y}_l}{l_0}\right|\right], \tag{B13}$$

where

$$\overline{K(q)} := Kq - g_A(Kq)^2. \tag{B14}$$

The boundary $\overline{K(q)}=0$ which delimits a positive from a negative effective CB gas temperature shrinks with increasing q. Hence order by order in powers of $h_1/2t$, the regime in which the fugacity expansion is well defined is controlled by the largest moment contributing to this order in Eq. (B13). We recognize in Eq. (B14) the scaling dimensions on the right hand side of Eq. (5.7).

3. CB gas interpretation of the fugacity expansion

Before going into a more detailed discussion of the fugacity expansion up to fourth order, we comment on some general features of the fugacity expansion and its close relationship to the replicated effective theory of section V. The impurity average over the two-point correlation function $\langle F_{12}\rangle$ in Eq. (4.2) can be recast as a summation over all possible configurations

$$\overline{\langle e^{\mathrm{i}\chi(\mathbf{x}_1)-\mathrm{i}\chi(\mathbf{x}_2)}\times e^{\mathrm{i}\chi(\mathbf{y}_1)+\cdots-\mathrm{i}\chi(\mathbf{y}_{2i})}\rangle^{\mathrm{unnor}}_{h_1=0}\times\langle e^{\mathrm{i}\chi(\mathbf{z}_1)+\cdots-\mathrm{i}\chi(\mathbf{z}_{2j})}\rangle^{\mathrm{unnor}}_{h_1=0}\times\cdots} \tag{B15}$$

of $2i+2j+\cdots$ vortices that screen the two external vortices at $\mathbf{x}_1$ and $\mathbf{x}_2$. The appropriate combinatorial weight that results from expanding the numerator and denominator in powers of the fugacity is not written here but can be found in the coming subsection. The first thermal average $\langle\cdots\rangle^{\mathrm{unnor}}_{h_1=0}$ comes from the expansion of the numerator in Eq. (3.15), while the remaining factors in $\langle\cdots\rangle^{\mathrm{unnor}}_{h_1=0}\times\cdots$ come from the expansion of the denominator. Notice that thermal averages are disconnected averages as is indicated by the superscript [see Eq. (B5)].

Introducing $\chi =: \chi' + \frac{it}{2\pi}\theta$ (since χ' and θ decouple), the above equation becomes

$$\overline{e^{-\frac{t}{2\pi}\theta(\mathbf{x}_1)+\cdots+\frac{t}{2\pi}\theta(\mathbf{z}_{2j})+\cdots}\langle e^{\mathrm{i}\chi'(\mathbf{x}_1)-\mathrm{i}\chi'(\mathbf{x}_2)}\times e^{\mathrm{i}\chi'(\mathbf{y}_1)+\cdots-\mathrm{i}\chi'(\mathbf{y}_{2i})}\rangle^{\mathrm{unnor}}_{h_1=0}\times\langle e^{\mathrm{i}\chi'(\mathbf{z}_1)+\cdots-\mathrm{i}\chi'(\mathbf{z}_{2j})}\rangle^{\mathrm{unnor}}_{h_1=0}\times\cdots}. \tag{B16}$$

Each term in the fugacity expansion can then be given the following CB gas interpretation. There is one charge Q, *the disorder charge*, associated with the field θ. It is also convenient to associate with each disconnected thermal average $\langle\cdots\rangle^{\mathrm{unnor}}_{h_1=0}$ appearing in (B16) a distinct thermal charge ε_a. More precisely, to order $(h_1/2t)^{2n}$ we introduce $n+1$ thermal charges ε_a, $a=0,\cdots,n$ (we take the thermal charges $\varepsilon_{m+1}=\cdots=\varepsilon_n=0$ to always vanish if there are only $m<n$ thermal factors, and the charges labeled by the subscript 0 always refer to the disconnected thermal average involving the two external charges at $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively). For example, if none of the coordinates in (B16) coincide we assign

- $(\varepsilon_0=+1,\varepsilon_1=0,\cdots,\varepsilon_n=0;Q=+1)$ to the CB charges at $\mathbf{x}_1$ and $\mathbf{y}_l$, $l=1,\cdots,i$.

- $(\varepsilon_0=-1,\varepsilon_1=0,\cdots,\varepsilon_n=0;Q=-1)$ to the CB charges at $\mathbf{x}_2$ and $\mathbf{y}_l$, $l=i+1,\cdots,2i$.

- $(\varepsilon_0 = 0, \varepsilon_1 = +1, \cdots, \varepsilon_n = 0; Q = +1)$ to the CB charges at $\mathbf{z}_l$, $l = 1, \cdots, j$.
- $(\varepsilon_0 = 0, \varepsilon_1 = -1, \cdots, \varepsilon_n = 0; Q = -1)$ to the CB charges at $\mathbf{z}_l$, $l = j+1, \cdots, 2j$.

Since coordinates belonging to *distinct* disconnected thermal averages can coincide (see the next subsection), we will also allow *complex or fused* charges of the form $(\varepsilon_0, \cdots, \varepsilon_n; Q)$ where

$$\sum_{a=0}^{n} \varepsilon_n = Q, \quad |\varepsilon_1| \le n, \quad |\varepsilon_2| \le n - |\varepsilon_1|, \cdots. \tag{B17}$$

It is then possible to systematically carry through the thermal and disorder average in (B16) by assuming that the *replicated* charges $(\varepsilon_0, \varepsilon_1, \cdots; Q)$ and $(\varepsilon'_0, \varepsilon'_1, \cdots; Q')$ are associated to operators with correlation functions given by

$$\left|\frac{\mathbf{w} - \mathbf{w}'}{a}\right|^{-2\pi g_A K^2 Q Q' + 2\pi K \sum_a \varepsilon_a \varepsilon'_a}. \tag{B18}$$

As we change the microscopic cut-off a and/or include randomness in h_1, two operators with charges $(\varepsilon_0, \varepsilon_1, \cdots; Q)$ and $(\varepsilon'_0, \varepsilon'_1, \cdots; Q')$, respectively, may fuse into one with charge $(\varepsilon_0 + \varepsilon'_0, \varepsilon_1 + \varepsilon'_1, \cdots; Q + Q')$. Thus, operators with arbitrary complex charges may appear in the fugacity expansion upon renormalization or averaging over random vortex fugacity. Although bare vortices in the XY model have a simple structure (being labeled by a single integer), it is striking to see that complex vortices *must* be accounted for in disorder averaged correlation functions in contrast to the clean limit (in the pure system these higher charges are always irrelevant). Complex vortices also appear formally in the replica approach (see section V). However, intuition is easily lost when taking the replica limit $r \downarrow 0$.

Our discussion of the fugacity expansion suggests that fugacities $Y_{(\varepsilon_0, \varepsilon_1, \cdots; Q)}$ should be introduced when operators with charge $(\varepsilon_0, \varepsilon_1, \cdots; Q)$ contribute to (B16). These fugacities, to a first approximation, are related to the density of *screening* charges of type $(\varepsilon_0, \varepsilon_1, \cdots; Q)$. Form invariance of the fugacity expansion under an infinitesimal rescaling of the short distance cutoff a:

$$a' := a e^{l}, \quad 0 < l \ll 1, \tag{B19}$$

would then imply that the fugacities obey RG equations. Since the scaling dimension of an operator depends on its charge, we expect the RG equations for the new fugacities to be different from each other. In particular, to order $2n$ in the fugacity, the generalized fugacities $Y_{(0,1,\cdots,1;Q)}, \cdots, Y_{(1,\cdots,1,0;Q)}$ are expected to be the most relevant operators within the family $Y_{(\varepsilon_0,\cdots,\varepsilon_n;Q)}$ [compare with Eq. (5.7)].

We will slightly abuse our notation in the following by using for the subscript of generalized fugacities the maximum number n of non-vanishing thermal charges (to order $2n$ in the fugacity expansion). For example, to fourth order, we will denote by $Y_{(1,1;2)}$ any of the three fugacities $Y_{(1,1,0;2)}$, $Y_{(1,0,1;2)}$, and $Y_{(0,1,1;2)}$ associated to the charges $(1,1,0;2)$, $(1,0,1;2)$, and $(0,1,1;2)$, respectively. Indeed, we will show that the RG equations for all three fugacities are the same.

We illustrate those general considerations with a detailed calculation of the fourth order correction to $\overline{\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle}$. We denote with $F^{(4)}_{12}$ the fourth order coefficient to the fugacity expansion (see the following subsection). It is then convenient to distinguish between three contributions to $\overline{F^{(4)}_{12}}$, denoted $\overline{A_{12}}$, $\overline{B_{12}}$, and $\overline{C_{12}}$, respectively. To fourth order in the fugacity expansion, we expect that complex vortices with charges $(\varepsilon_0, \varepsilon_1, \varepsilon_2; Q)$ emerge. Correspondingly, it should be possible to recast the RG equations in terms of fugacities $Y_{(\varepsilon_0,\varepsilon_1,\varepsilon_2;Q)}$. This is indeed so.

The contribution $\overline{A_{12}}$ is nothing but the second order contribution squared. Hence, it is due to the screening of two external vortices with charges $(+1;+1)$ and $(-1;-1)$, respectively, by four thermal vortices with charges of type $(\pm 1; \pm 1)$. It is given by:

$$\overline{A_{12}} \left(\frac{h_1}{2t}\right)^4 \approx \left|\frac{\mathbf{x}_{12}}{a}\right|^{-2\pi \bar{K}} \times \frac{1}{2}\left[2\pi x^{(2)} \ln\left|\frac{\mathbf{x}_{12}}{a}\right|\right]^2, \tag{B20}$$

where $x^{(2)}$ is the second order correction of Eq. (4.5) when $q = 1$:

$$x^{(2)} := -4\pi^3 \left[\bar{K}^2 - K^4 g_A^2\right] Y^2_{(1;1)} \int_1^{L/a} dy y^{3-2\pi\bar{K}}. \tag{B21}$$

This is an encouraging result since it justifies a posteriori our assumption in Eq. (4.4) that logarithmic corrections can be reexponentiated.

The contribution $\overline{B_{12}}$ is due to the screening of our two external vortices by either three replicated vortices with charges of type $(+1,+1;+2)$ (fused), $(-1,0;-1)$ and $(0,-1;-1)$, respectively, or three replicated vortices with charges $(+1,-1;0)$ (fused), $(-1,0;-1)$ and $(0,+1;+1)$, respectively.

The term $\overline{C_{12}}$ is induced by the screening of two external vortices at $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, by either two fused vortices with charges of type $(\pm 1, \mp 1; 0)$, or by two fused vortices with charges of type $(\pm 1, \pm 1; \pm 2)$. Concretely, we find

$$\overline{C_{12}} \left(\frac{h_1}{2t}\right)^4 \approx \left|\frac{\mathbf{x}_{12}}{a}\right|^{-2\pi \bar{K}} \times 2\pi x^{(4)}_c \ln\left|\frac{\mathbf{x}_{12}}{a}\right|, \tag{B22}$$

where $x^{(4)}_c$ is the fourth order correction due to two-body renormalization effects:

$$x^{(4)}_c := -4\pi^3 K^2 Y^2_{(1,-1;0)} \int_1^{L/a} dy y^{3-4\pi K} \tag{B23}$$

$$+4\pi^3\left\{4K^4g_A^2-\frac{\left[\overline{K(2)}\right]^2}{4}\right\}Y^2_{(1,1;2)}\int\limits_1^{L/a}dyy^{3-2\pi\overline{K(2)}}.$$

The first line on the right hand side of Eq. (B23) is fully oblivious of the disorder. The second line is dramatically sensitive to it since it yields a new boundary for analyticity of the fugacity expansion. The two lines are equal in the absence of disorder.

Furthermore, the second line on the right hand side of Eq. (B23) implies that, in contrast to the second order contribution to the fugacity expansion of the mean $\overline{\langle F_{12}\rangle}$, the scaling exponent $2\pi\overline{K(2)}$ of $\overline{\langle F_{12}\rangle^2_{h_1=0}}$ enters in the fourth order correction to the scaling exponent $2\pi\bar{K}$ of $\overline{\langle F_{12}\rangle_{h_1=0}}$. Hence, the second moment of $\langle F_{12}\rangle$ couples to the mean of $\langle F_{12}\rangle$ in the RG flow to fourth order in the fugacity.

From the scaling dimension $2\pi K-4\pi g_AK^2=\pi\overline{K(2)}$ of the operator with charge of type $(1,1;2)$, we find the RG equation (4.16) of $Y_{(1,1;2)}$. Equation (B23) tells us that $Y_{(1,1;2)}$ indeed enters the RG equation of $\bar{K}$. Hence, the perturbative RG equations breaks down when $2\pi\overline{K(2)}<4$.

The pattern for renormalization should become clear from our fourth order calculation. To each new order $2n$ in $h_1/2t$ we need to introduce new fugacities for replicated vortices labeled by their thermal and disorder charges

$$(\varepsilon_0,\cdots,\varepsilon_n;\sum_{a=1}^n\varepsilon_a),\quad \varepsilon_a=\pm1,\quad a=0,\cdots,n,$$

to close the RG equations. The scaling dimension of the fugacity

$$Y_{(\varepsilon_1,\cdots,\varepsilon_n;\sum_{a=1}^n\varepsilon_a)}$$

on the critical plane of Fig. 1 is deduced from that of

$$\begin{aligned}&\exp\left[\mathrm{i}\sum_{a=1}^n\varepsilon_a\chi_a(\mathbf{y})\right]=\\&\exp\left[\mathrm{i}\sum_{a=1}^n\varepsilon_a\chi'_a(\mathbf{y})\right]\exp\left[-2\pi K\theta(\mathbf{y})\sum_{a=1}^n\varepsilon_a\right].\end{aligned}\qquad\text{(B24)}$$

For any infinitesimal value of the disorder strength g_A, the contribution to the coefficient of the fugacity expansion to order $2n$ that defines the regime of analyticity [see Eq. (4.14)] describes the screening of the CB interaction by two tightly bound fused vortices of charges $(\varepsilon,\cdots,\varepsilon;n\varepsilon)$, $\varepsilon=\pm1$, respectively. The scaling with system size of this coefficient is thus given by Eq. (4.13). Subleading contributions to the coefficient of the expansion originate from screening of the CB interaction by three and more replicated vortices.

Finally, we would like to stress that there exists a one to one correspondence between the replicated vortices appearing in the fugacity expansion and the replicated primary fields constructed in section V [see Eqs. (5.5,5.6)] as was first suggested by Korshunov [21].

D. Two-point function $\overline{\langle F_{\mathbf{x}_1,\mathbf{x}_2}\rangle}$ up to order $(h_1/2t)^4$

It is very instructive to carry the fugacity expansion of the two-point function

$$F_{\mathbf{x}_1,\mathbf{x}_2}:=e^{\mathrm{i}\chi(\mathbf{x}_1)-\mathrm{i}\chi(\mathbf{x}_2)}\equiv F_{12}.\qquad\text{(B25)}$$

This calculation is the crucial ingredient in the RG analysis of the KT transition.

For the pure case quasi-long-range order holds if F_{12} decays algebraically with separation with a scaling exponent $\kappa\equiv x(1)$ [see Eq. (4.4)] which is an analytic function of fugacity Y_1 in the vicinity of $Y_1=0$. The transition temperature to a disordered phase is deduced from the boundary along the line $1/K\geq0$ for which the scaling exponent κ becomes non-analytic.

In the presence of disorder, Rubinstein et al. performed the same analysis after averaging over disorder the fugacity expansion of F_{12} term by term up to second order in Y_1. They inferred the parabolic boundary of Fig. 1 from the onset of a non-analytic dependency of κ on fugacity Y_1. We repeat their argument and show, in the spirit of Korshunov's analysis [21], how higher moments of correlation functions invalidate their conclusion beyond second order in the fugacity expansion.

The power expansion in $Y_1\propto h_1/2t$ is given by

$$\langle F_{12}\rangle:=\sum_{n=0}^\infty F_{12}^{(n)}\ (h_1/2t)^n:=\frac{\sum_{m=0}^\infty f_{12}^{(m)}\ (h_1/2t)^m}{1+\sum_{n=1}^\infty Z^{(n)}\ (h_1/2t)^n},\qquad\text{(B26)}$$

where $F_{12}^{(2n+1)}=f_{12}^{(2n+1)}=Z^{(2n+1)}=0$ and

$$\begin{aligned}F_{12}^{(0)}&=f_{12}^{(0)},\\F_{12}^{(2)}&=f_{12}^{(2)}-f_{12}^{(0)}Z^{(2)},\\F_{12}^{(4)}&=f_{12}^{(4)}-\left[f_{12}^{(2)}Z^{(2)}+f_{12}^{(0)}Z^{(4)}\right]+f_{12}^{(0)}Z^{(2)}\times Z^{(2)},\end{aligned}\qquad\text{(B27)}$$

up to fourth order in $h_1/2t$. The coefficients in the power expansions in $h_1/2t$ of the numerator and denominator are

$$f_{12}^{(2n)} = \frac{1}{(n!)^2} \int \underbrace{d^2\mathbf{y}_1 \cdots d^2\mathbf{y}_{2n}}_{\neq} \left\langle e^{\mathrm{i}\chi(\mathbf{y}_1)+\cdots+\mathrm{i}\chi(\mathbf{y}_n)-\mathrm{i}\chi(\mathbf{y}_{n+1})-\cdots-\mathrm{i}\chi(\mathbf{y}_{2n})+\mathrm{i}\chi(\mathbf{x}_1)-\mathrm{i}\chi(\mathbf{x}_2)} \right\rangle_{h_1=0}^{\mathrm{unnor}}, \tag{B28}$$

$$Z^{(2n)} = \frac{1}{(n!)^2} \int \underbrace{d^2\mathbf{y}_1 \cdots d^2\mathbf{y}_{2n}}_{\neq} \left\langle e^{\mathrm{i}\chi(\mathbf{y}_1)+\cdots+\mathrm{i}\chi(\mathbf{y}_n)-\mathrm{i}\chi(\mathbf{y}_{n+1})-\cdots-\mathrm{i}\chi(\mathbf{y}_{2n})} \right\rangle_{h_1=0}^{\mathrm{unnor}}, \tag{B29}$$

respectively. Since we are assuming the existence of a dipole phase, thermal vortices are taken with a hardcore as is indicated by the constraint that the coordinates of the vortices cannot coincide. Hence, we are implicitly using a short distance cutoff a for the thermal vortices. As a matter of principle, this cutoff need not be the same as that used at short distances for the quenched vortices. Nevertheless, for notational simplicity, we will assume that quenched vortices share the same hardcore radius.

To lowest order in $Y_1 \propto h_1/2t$:

$$\overline{F_{12}^{(0)}} = \left| \frac{\mathbf{x}_1 - \mathbf{x}_2}{a} \right|^{-2\pi\bar{K}} \equiv \left| \frac{\mathbf{x}_{12}}{a} \right|^{-2\pi\bar{K}}. \tag{B30}$$

To second order in $Y_1 \propto h_1/2t$:

$$\begin{aligned}
\overline{F_{12}^{(2)}} =& \left| \frac{a^2}{\mathbf{x}_1 - \mathbf{x}_2} \right|^{2\pi\bar{K}} \int \underbrace{d^2\mathbf{y}_1 d^2\mathbf{y}_2}_{\neq} \frac{\mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi\bar{K}) - \mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi K^2 g_A)}{|\mathbf{y}_1 - \mathbf{y}_2|^{2\pi\bar{K}}} \\
\equiv& \left| \frac{a^2}{\mathbf{x}_{12}} \right|^{2\pi\bar{K}} \underbrace{\int_{12}}_{\neq} \frac{\mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi\bar{K}) - \mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi K^2 g_A)}{|\mathbf{y}_{12}|^{2\pi\bar{K}}} \\
\approx& |\mathbf{x}_{12}/a|^{-2\pi\bar{K}} \times 8\pi^4 \left(\bar{K}^2 - K^4 g_A^2\right) \times \left(\int_a^L d|\mathbf{y}_{12}|\ |\mathbf{y}_{12}|^3 |\mathbf{y}_{12}/a|^{-2\pi\bar{K}} \right) \times \ln|\mathbf{x}_{12}/a|,
\end{aligned} \tag{B31}$$

where

$$\mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;x) := \left[\frac{|\mathbf{y}_1 - \mathbf{x}_1||\mathbf{y}_2 - \mathbf{x}_2|}{|\mathbf{y}_1 - \mathbf{x}_2||\mathbf{y}_2 - \mathbf{x}_1|} \right]^x. \tag{B32}$$

If we interpret the presence of a logarithmic correction on the right hand side of Eq. (B31) as the first term in the expansion of

$$\overline{F_{12}} = \left| \frac{a}{\mathbf{x}_{12}} \right|^{\kappa} = \left| \frac{a}{\mathbf{x}_{12}} \right|^{2\pi\bar{K}+\delta\kappa} \tag{B33}$$

in powers of $\delta\kappa$, then the scaling exponent κ governing the algebraic decay of $\overline{F_{12}}$ is given by

$$\kappa = 2\pi\bar{K} - 8\pi^4 \left(\bar{K}^2 - K^4 g_A^2\right) \left(\int_1^{\frac{L}{a}} dx\ x^{3-2\pi\bar{K}} \right) \left(a^4 Y_1^2\right) + \mathcal{O}(a^8 Y_1^4). \tag{B34}$$

We see that the condition that the scaling exponent for the algebraic decay of the two-point function be analytic in the fugacity Y_1 in the vicinity of vanishing fugacity yields the parabolic boundary in Fig. 1.

To fourth order in $Y_1 \propto h_1/2t$ we must distinguish between three different effects present *both with or without disorder*. Indeed, we can write

$$F_{12}^{(4)} = A_{12} + B_{12} + C_{12}. \tag{B35}$$

Renormalizations of the interaction between two external vortices due to: i) four-body effects, ii) three-body effects, iii) two-body effects, are denoted by A_{12}, B_{12}, and C_{12}, respectively. Three-body and two-body effects must be accounted for when the separation $|\mathbf{x}_{12}|$ between the two external charges is much larger than the hardcore radius a, and when the separation between vortices is within the hardcore radius. Formally, we implement these renormalization effects

whenever coordinates of vortices coincide in the integrands. This is never allowed due to the hardcore constraint for both $\overline{f_{12}^{(4)}}$ and $\overline{f_{12}^{(0)} Z^{(4)}}$. However, coordinates can coincide when expanding the inverse partition function in Eq. (B26). For example,

$$
\begin{aligned}
\overline{f_{12}^{(2)} Z^{(2)}} = & \int \underbrace{d^2\mathbf{y}_1 d^2\mathbf{y}_3}_{\neq} \int \underbrace{d^2\mathbf{y}_2 d^2\mathbf{y}_4}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_2)-\chi(\mathbf{y}_4)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
= & \underbrace{\int_{1234}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_2)-\chi(\mathbf{y}_4)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{134}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_4)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{132}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_2)-\chi(\mathbf{y}_1)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{134}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_3)-\chi(\mathbf{y}_4)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{132}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_2)-\chi(\mathbf{y}_3)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{13}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)]}\right\rangle_{h_1=0}^{\text{unnor}}} \\
+ & \underbrace{\int_{13}}_{\neq} \overline{\left\langle e^{\mathrm{i}[\chi(\mathbf{y}_1)-\chi(\mathbf{y}_3)+\chi(\mathbf{x}_1)-\chi(\mathbf{x}_2)]}\right\rangle_{h_1=0}^{\text{unnor}} \times \left\langle e^{\mathrm{i}[\chi(\mathbf{y}_3)-\chi(\mathbf{y}_1)]}\right\rangle_{h_1=0}^{\text{unnor}}} .
\end{aligned}
\tag{B36}
$$

Four-body renormalization effects are

$$
\begin{aligned}
&\overline{A_{12}} = \\
&\frac{1}{(2!)^2}\underbrace{\int_{1234}}_{\neq}\left[\frac{a}{|\mathbf{x}_{12}|}\times\frac{a^2|\mathbf{y}_{12}||\mathbf{y}_{34}|}{|\mathbf{y}_{13}||\mathbf{y}_{14}||\mathbf{y}_{23}||\mathbf{y}_{24}|}\times\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_1|}\right]^{2\pi\bar{K}} - \\
&\frac{1}{(2!)^2}\underbrace{\int_{3412}}_{\neq}\left[\frac{a}{|\mathbf{x}_{12}|}\frac{a^2|\mathbf{y}_{12}||\mathbf{y}_{34}|}{|\mathbf{y}_{13}||\mathbf{y}_{14}||\mathbf{y}_{23}||\mathbf{y}_{24}|}\right]^{2\pi\bar{K}}\left[\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_1|}\right]^{+2\pi K^2 g_A} + \\
&\underbrace{\int_{3412}}_{\neq}\left[\frac{a}{|\mathbf{x}_{12}|}\frac{a^2}{|\mathbf{y}_{13}||\mathbf{y}_{24}|}\right]^{2\pi\bar{K}}\times\left[\frac{|\mathbf{y}_{12}||\mathbf{y}_{34}|}{|\mathbf{y}_{14}||\mathbf{y}_{32}|}\times\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_1|}\right]^{+2\pi K^2 g_A} - \\
&\underbrace{\int_{1432}}_{\neq}\left[\frac{a}{|\mathbf{x}_{12}|}\frac{a^2}{|\mathbf{y}_{13}||\mathbf{y}_{24}|}\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{2\pi\bar{K}}\left[\frac{|\mathbf{y}_{12}||\mathbf{y}_{34}|}{|\mathbf{y}_{14}||\mathbf{y}_{32}|}\frac{|\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_1|}\right]^{+2\pi K^2 g_A} .
\end{aligned}
\tag{B37}
$$

Assuming screening by a dilute gas of dipoles reduces the integration over the coordinates $\mathbf{y}_1, \cdots, \mathbf{y}_4$ of the four thermal vortices to

$$\overline{A_{12}} \approx 2 \times \frac{1}{(2!)^2} \left|\frac{a^2}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \left\{ \underbrace{\int_{12}}_{\neq} \frac{\mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi\bar{K}) - \mathcal{K}_{\mathbf{x}_1\mathbf{x}_2}(\mathbf{y}_1,\mathbf{y}_2;2\pi K^2 g_A)}{|\mathbf{y}_1-\mathbf{y}_2|^{2\pi\bar{K}}} \right\}^2$$

$$= \frac{1}{2}\left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \left\{ 8\pi^4\left(\bar{K}^2 - K^4 g_A^2\right)\left(\int_a^L d|\mathbf{y}_{12}|\ |\mathbf{y}_{12}|^3 \left|\frac{\mathbf{y}_{12}}{a}\right|^{-2\pi\bar{K}}\right) \ln\left|\frac{\mathbf{x}_{12}}{a}\right| \right\}^2. \tag{B38}$$

Three-body renormalization effects are

$$\overline{B_{12}} = \tag{B39}$$

$$2\underbrace{\int_{134}}_{\neq} \left[\frac{a}{|\mathbf{x}_{12}|}\right]^{2\pi\bar{K}} \times \left[\frac{a^2}{|\mathbf{y}_{13}||\mathbf{y}_{14}|}\right]^{2\pi K} \times \left[\frac{a^4|\mathbf{y}_{34}|}{|\mathbf{y}_{14}|^2|\mathbf{y}_{13}|^2} \times \frac{|\mathbf{y}_1-\mathbf{x}_1|^2|\mathbf{y}_3-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2|^2|\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_1|}\right]^{-2\pi K^2 g_A} -$$

$$2\underbrace{\int_{134}}_{\neq} \left[\frac{a}{|\mathbf{x}_{12}|}\right]^{2\pi\bar{K}} \times \left[\frac{a^2|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_{13}||\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_{14}|}\right]^{2\pi K} \times$$

$$\left[\frac{a^4|\mathbf{y}_{34}|}{|\mathbf{y}_{14}|^2|\mathbf{y}_{13}|^2} \times \frac{|\mathbf{y}_1-\mathbf{x}_1|^2|\mathbf{y}_3-\mathbf{x}_2||\mathbf{y}_4-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2|^2|\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_4-\mathbf{x}_1|}\right]^{-2\pi K^2 g_A} +$$

$$2\underbrace{\int_{132}}_{\neq} \left[\frac{a}{|\mathbf{x}_{12}|}\right]^{2\pi\bar{K}} \times \left[\frac{a^2}{|\mathbf{y}_{13}||\mathbf{y}_{21}|}\right]^{2\pi K} \times \left[\frac{a}{|\mathbf{y}_{32}|} \times \frac{|\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{-2\pi K^2 g_A} -$$

$$2\underbrace{\int_{132}}_{\neq} \left[\frac{a}{|\mathbf{x}_{12}|}\right]^{2\pi\bar{K}} \times \left[\frac{a^2|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_{13}||\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1||\mathbf{y}_{21}|}\right]^{2\pi K} \times \left[\frac{a}{|\mathbf{y}_{32}|} \times \frac{|\mathbf{y}_2-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_2-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{-2\pi K^2 g_A}.$$

After introducing the center of mass coordinates,

$$\mathbf{Y} := \frac{1}{3}\left(\mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3\right), \tag{B40}$$

$$\mathbf{y}_{12} := \mathbf{y}_1 - \mathbf{y}_2, \tag{B41}$$

$$\mathbf{y}_{13} := \mathbf{y}_1 - \mathbf{y}_3, \tag{B42}$$

it is possible to show that

$$\begin{aligned}
\overline{B_{12}}\left(\frac{h_1}{2t}\right)^4 \approx& \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| (2\pi)^3 \times 2\left(K^4 g_A^2 - \bar{K}^2\right) Y_{(+1,+1;+2)} Y_{(-1,0;-1)} Y_{(0,-1;-1)} \frac{\tilde{S}^{(11)}}{2} \\
&+ \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| (2\pi)^3 \times 2\left(K^4 g_A^2 - \bar{K}^2\right) Y_{(+1,+1;+2)} Y_{(-1,0;-1)} Y_{(0,-1;-1)} \frac{\tilde{S}^{(21)}}{2} \\
&+ \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| (2\pi)^3 \times 4K^3 g_A Y_{(+1,-1;0)} Y_{(-1,0;-1)} Y_{(0,+1;+1)} \frac{\tilde{S}^{(13)}}{2} \\
&- \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| (2\pi)^3 \times 4K^3 g_A Y_{(+1,-1;0)} Y_{(-1,0;-1)} Y_{(0,+1;+1)} \frac{\tilde{S}^{(23)}}{2},
\end{aligned} \tag{B43}$$

where we have defined four dimensionless integrals

$$\tilde{S}^{(11)} := \int\limits_{|\mathbf{y}_{12}|>a} \frac{d^2\mathbf{y}_{12}}{a^2} \int\limits_{|\mathbf{y}_{13}|>a} \frac{d^2\mathbf{y}_{13}}{a^2} \left|\frac{a}{\mathbf{y}_{12}}\right|^{2\pi\overline{K(2)/2}} \left|\frac{\mathbf{y}_{13}-\mathbf{y}_{12}}{a}\right|^{-2\pi K^2 g_A} \left|\frac{a}{\mathbf{y}_{13}}\right|^{2\pi\overline{K(2)/2}} \frac{\mathbf{y}_{12}\cdot\mathbf{y}_{12}}{a^2}, \tag{B44}$$

$$\tilde{S}^{(13)} := \int\limits_{|\mathbf{y}_{12}|>a} \frac{d^2\mathbf{y}_{12}}{a^2} \int\limits_{|\mathbf{y}_{13}|>a} \frac{d^2\mathbf{y}_{13}}{a^2} \left|\frac{a}{\mathbf{y}_{12}}\right|^{2\pi\overline{K(2)/2}} \left|\frac{\mathbf{y}_{13}-\mathbf{y}_{12}}{a}\right|^{-2\pi K^2 g_A} \left|\frac{a}{\mathbf{y}_{13}}\right|^{2\pi\overline{K(2)/2}} \frac{\mathbf{y}_{12}\cdot\mathbf{y}_{13}}{a^2}, \tag{B45}$$

$$\tilde{S}^{(21)} := \int\limits_{|\mathbf{y}_{12}|>a} \frac{d^2\mathbf{y}_{12}}{a^2} \int\limits_{|\mathbf{y}_{13}|>a} \frac{d^2\mathbf{y}_{13}}{a^2} \left|\frac{a}{\mathbf{y}_{12}}\right|^{2\pi K} \left|\frac{\mathbf{y}_{13}-\mathbf{y}_{12}}{a}\right|^{+2\pi K^2 g_A} \left|\frac{a}{\mathbf{y}_{13}}\right|^{2\pi K} \frac{\mathbf{y}_{12}\cdot\mathbf{y}_{12}}{a^2}, \tag{B46}$$

$$\tilde{S}^{(23)} := \int\limits_{|\mathbf{y}_{12}|>a} \frac{d^2\mathbf{y}_{12}}{a^2} \int\limits_{|\mathbf{y}_{13}|>a} \frac{d^2\mathbf{y}_{13}}{a^2} \left|\frac{a}{\mathbf{y}_{12}}\right|^{2\pi K} \left|\frac{\mathbf{y}_{13}-\mathbf{y}_{12}}{a}\right|^{+2\pi K^2 g_A} \left|\frac{a}{\mathbf{y}_{13}}\right|^{2\pi K} \frac{\mathbf{y}_{12}\cdot\mathbf{y}_{13}}{a^2}. \tag{B47}$$

Here, we have introduced the dimensionless fugacities $Y_{(\varepsilon_1,\varepsilon_2;\varepsilon_1+\varepsilon_2)}$, $Y_{(0,\varepsilon;\varepsilon)}$, and $Y_{(\varepsilon,0;\varepsilon)}$, where $\varepsilon_1=\varepsilon_2=\pm1$ and $\varepsilon=\pm1$, respectively. Their bare values are given by $\left(\frac{a^2 h_1}{2t}\right)^{4/3}$. Whereas under a RG rescaling $a=e^l a'$ with $0<l\ll1$, the Y's and product thereof are renormalized multiplicatively, an open problem is to find numerical factors entering additive renormalization effect due to three-body effects. This is difficult in this real space RG approach since we must perform four-dimensional integrals with complicated integrands and intricate boundaries.

Form invariance under an infinitesimal rescaling of the short distance cutoff a: $a':=ae^l$, $0<l\ll1$, of $\overline{B_{12}}$ implies that the three fugacities ($\varepsilon_1,\varepsilon_2=\pm1$)

$$Y_{(\varepsilon_1,\varepsilon_2;\varepsilon_1+\varepsilon_2)}, \quad Y_{(-\varepsilon_1,0;-\varepsilon_1)}, \quad Y_{(0,-\varepsilon_2;-\varepsilon_2)}, \tag{B48}$$

obey:

1. They always appear in the combinations

$$Y_{(\varepsilon_1,\varepsilon_2;\varepsilon_1+\varepsilon_2)} \times Y_{(-\varepsilon_1,0;-\varepsilon_1)} \times Y_{(0,-\varepsilon_2;-\varepsilon_2)}. \tag{B49}$$

2. Those combinations renormalize multiplicatively according to two rules ($\varepsilon=\pm1$)

$$\begin{aligned}\left[Y_{(\varepsilon,\varepsilon;2\varepsilon)}Y_{(-\varepsilon,0;-\varepsilon)}Y_{(0,-\varepsilon;-\varepsilon)}\right]' &= Y_{(\varepsilon,\varepsilon;2\varepsilon)}Y_{(-\varepsilon,0;-\varepsilon)}Y_{(0,-\varepsilon;-\varepsilon)}\ e^{6-2\pi\overline{K(2)}-2\pi K^2 g_A},\\ \left[Y_{(\varepsilon,-\varepsilon;0)}Y_{(-\varepsilon,0;-\varepsilon)}Y_{(0,+\varepsilon;+\varepsilon)}\right]' &= Y_{(\varepsilon,-\varepsilon;0)}Y_{(-\varepsilon,0;-\varepsilon)}Y_{(0,+\varepsilon;+\varepsilon)}\ e^{6-4\pi K+2\pi K^2 g_A}.\end{aligned}$$

Additionally, if we require the individual multiplicative renormalization rules

$$Y'_{(\varepsilon,\varepsilon;2\varepsilon)} = Y_{(\varepsilon,\varepsilon;2\varepsilon)} e^{l[2-\pi\overline{K(2)}]}, \tag{B50}$$

$$Y'_{(\varepsilon,-\varepsilon;0)} = Y_{(\varepsilon,-\varepsilon;0)} e^{l[2-2\pi K]}, \tag{B51}$$

$$Y'_{(\varepsilon,0;\varepsilon)} = Y_{(\varepsilon,0;\varepsilon)} e^{l[2-\pi\bar{K})]}, \tag{B52}$$

$$Y'_{(0,\varepsilon;\varepsilon)} = Y_{(0,\varepsilon;\varepsilon)} e^{l[2-\pi\bar{K}]}, \tag{B53}$$

we can summarize Eqs. (B50,B50) by

$$Y_{(+1,+1;2)} \times Y_{(-1,0;-1)} \times Y_{(0,-1;-1)} \sim e^{\mathrm{i}(\chi_1+\chi_2)(\mathbf{y}_1)} \times e^{-\mathrm{i}\chi_1(\mathbf{y}_2)} \times e^{-\mathrm{i}\chi_2(\mathbf{y}_3)}, \tag{B54}$$

$$Y_{(+1,-1;0)} \times Y_{(-1,0;-1)} \times Y_{(0,+1;+1)} \sim e^{\mathrm{i}(\chi_1-\chi_2)(\mathbf{y}_1)} \times e^{-\mathrm{i}\chi_1(\mathbf{y}_2)} \times e^{+\mathrm{i}\chi_2(\mathbf{y}_3)}. \tag{B55}$$

Equations (B54,B55) tell us that any of the three thermal vortices renormalizing in $\overline{B_{12}}$ the CB interaction between two external vortices can be thought of as some *local* linear combination of the two replicated real scalar fields $\chi_{1,2}$. We remember that $\chi_{1,2}$ couple to the disorder in such a way that the fields $\chi'_{1,2}$ defined by $\chi_{1,2}=\chi'_{1,2}+\mathrm{i}2\pi K\theta$ are free scalar fields independent of the disorder θ. The fugacity $Y_{(\varepsilon_1,\varepsilon_2;\varepsilon_1+\varepsilon_2)}$ is thus labeled by three charges (measured in the appropriate units):

- The thermal charge ε_1 of χ'_1.
- The thermal charge ε_2 of χ'_2.
- The disorder charge $\varepsilon_1+\varepsilon_2$ of θ.

Finally, two-body renormalization effects are

$$\overline{C_{12}} = \underbrace{\int_{13}}_{\neq} \left[\frac{a}{|\mathbf{x}_{12}|}\right]^{2\pi\bar{K}} \times \tag{B56}$$

$$\left\{ \left[\frac{a}{|\mathbf{y}_{13}|}\right]^{2\pi\overline{K(2)}} \times \left[\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{-2\times 2\pi K^2 g_A} + \left[\frac{a}{|\mathbf{y}_{13}|}\right]^{2\times 2\pi K} - \right.$$

$$\left. \left[\frac{a}{|\mathbf{y}_{13}|}\right]^{2\pi\overline{K(2)}} \times \left[\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{2\pi K-2\times 2\pi K^2 g_A} - \left[\frac{a}{|\mathbf{y}_{13}|}\right]^{2\times 2\pi K} \times \left[\frac{|\mathbf{y}_1-\mathbf{x}_1||\mathbf{y}_3-\mathbf{x}_2|}{|\mathbf{y}_1-\mathbf{x}_2||\mathbf{y}_3-\mathbf{x}_1|}\right]^{2\pi K} \right\}.$$

The major difference between $\overline{C_{12}}$ and Eq. (B31) is the exponent of the coordinate $\mathbf{y}_{13}$ given by $\overline{K(2)} = 2K - 4K^2 g_A$. It is possible to show that

$$\overline{C_{12}} \left(\frac{h_1}{2t}\right)^4 \approx \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| 8\pi^4 \left\{ 4K^4 g_A^2 - \frac{\left[\overline{K(2)}\right]^2}{4} \right\} Y^2_{(1,1;2)} \int_1^{\frac{L}{a}} dy y^{3-2\pi\overline{K(2)}}$$

$$- \left|\frac{a}{\mathbf{x}_{12}}\right|^{2\pi\bar{K}} \ln\left|\frac{\mathbf{x}_{12}}{a}\right| 8\pi^4 K^2 Y^2_{(1,-1;0)} \int_1^{\frac{L}{a}} dy y^{3-4\pi K}. \tag{B57}$$

Here, we have introduced the dimensionless fugacities $Y_{(1,1;2)}$ and $Y_{(1,-1;0)}$ whose bare values are equal to $\left(\frac{a^2 h_1}{2t}\right)^2$.

In the absence of disorder, renormalization effects due to higher charge vortices do not modify the boundary extracted from Eq. (B31). This is nothing but the statement that $\cos(\chi)$ is the most relevant operator of the family $\cos(\chi_1)\times\cdots\times\cos(\chi_q), \cdots, \cos(q\chi)$, $q\in\mathbf{N}$, along the Gaussian fixed line $1/K \geq 0$, $g_A = 0$, $Y_1 = 0$. However, in the presence of the random vector potential $\tilde{\partial}_\mu\theta$ the relevance of $\cos(\chi_1)\times\cdots\times\cos(\chi_q)$ increases with $q\in\mathbf{N}$.

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